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Methods of Dynamic and Nonsmooth Optimization

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Preface

This monograph is based on a series of lectures given at a Regional Conference held at Emory University, Atlanta, Georgia, in 1986. The subjects treated here have been in an active state of development, and accordingly I have incorporated more recent results in the final version of this work.

I wish to thank Professors D. Ford and L. Riddle for initiating and hosting the conference, and for making it so pleasant an event. In addition, I take this opportunity to gratefully acknowledge the substantial contributions of my colleagues Philip Loewen and Richard Vinter to the material covered here. My thanks and admiration also go to Louise Letendre for so quickly becoming a TEXpert, and for her fine work on the manuscript.

Finally, I dedicate this book to my wife, Gail, who has brought love, happiness, and definition to my life, and who put up with my finishing it on what should have been our honeymoon.

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CHAPTER 1

Nonsmooth Analysis and Geometry

1.1. Introduction.

The term "nonsmooth analysis" was coined[1] to convey the idea of a systematic study and application of the differential properties of functions (and sets) not differentiable (or "tangentializable") in the usual sense. Together with a growing body of theory and applications, the passage of time has led to a widespread acceptance of the interest and utility inherent in such a theory. This is not surprising, perhaps, for the derivative is such a basic tool of analysis that its systematic extension to new domains is a natural goal. And it should not be surprising that optimization has been the primary impetus in this regard, if we consider the central role played by the calculus of variations in the development of functional analysis. At present, the range of successful applications of nonsmooth analysis demonstrates that it does not constitute mere generalization for its own sake, but provides a useful tool, and indeed a new point of view, in the study of many different issues in optimization and analysis.

The central elements in the nonsmooth calculus that we propose to survey in this chapter are the (generalized) normal cone and tangent cone to a closed set and the (generalized) subdifferential and directional derivative of a lower semicontinuous function, all of which originated in the author's initial work in the area [C1973], [C1975a]. Many investigators have since contributed to the development of these concepts, and a number of extensions have been made. An inescapable conclusion, however, is that the part of the theory that we shall discuss here lies in the core of the subject and constitutes a worthwhile investment of time for those seeking to learn about nonsmooth analysis. After reviewing the theory, we shall apply it, in a particularly accessible context, to the differential analysis of value functions. These functions form an important class of

[1] The term was coined by the author in the 1970s; the first published reference to this term appears to be in [C1978a].

nonsmooth functions arising in optimization, and they play a central role in later chapters.

The following provides an example of a value function. Define $V(\alpha)$, for a given value of α, to be the minimum over all x of $f(x)$ subject to $h(x) = \alpha$. (We do not make precise here the spaces and hypotheses used.) A moment's thought will confirm that for any x, by definition of V, we have

$$V(h(x)) \leq f(x).$$

Suppose now that $\hat{x}$ solves the following problem:

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } \ h(x) = 0. \end{aligned}$$

Then, again by definition, we have $V(0) = V(h(\hat{x})) = f(\hat{x})$. Combining these two observations, we conclude that $\hat{x}$ is an unconstrained minimizer for the function $x \longrightarrow f(x) - V(h(x))$. Setting the derivative at $\hat{x}$ equal to 0 gives

$$f'(\hat{x}) - V'(0)h'(\hat{x}) = 0,$$

which (with $\lambda = -V'(0)$) is the appropriate version of the well-known Lagrange multiplier rule for the problem that $\hat{x}$ solves.

We have not seen this particular proof of the multiplier rule anywhere, probably because it has a fatal flaw: the implicit assumption that V is differentiable. Indeed, even when f and h are smooth, V may not be. Yet the "proof" given above is useful in suggesting the value of a generalized calculus for nondifferentiable functions on the one hand, as well as the relationship between the Lagrange multipliers appearing in the necessary conditions and the sensitivity of the problem to perturbations in the constraints (as measured by rates of change of V) on the other. These themes are important ones in this chapter.

1.2. A geometrical starting point: Proximal normals.

In recent years, great emphasis has been placed on the analytical theory of nonsmooth analysis. But one of the points we shall make in this monograph is the interest in focusing more upon the geometrical nature of the theory. Our starting point is the *perpendicular*, a simple geometrical notion that was actually at the heart of our original approach, was somewhat ignored for some time, and that is now emerging as potentially the most powerful and simultaneously the simplest basic tool of nonsmooth analysis.

Let C be a nonempty closed subset of $\mathbb{R}^n$, and let x be a point in $\mathbb{R}^n$ not lying in C. Then there is at least one point c in C that is closest in C to x, i.e., such that

$$|x - c'| \geq |x - c| \quad \text{for all} \quad c' \in C. \tag{1.1}$$

Thus the open ball of radius $|x-c|$ centered at x contains no points of C, but c is in its boundary. We can rewrite (1.1) as follows. Square both sides and express them in terms of inner products to get

$$\langle x - c', x - c' \rangle \geq \langle x - c, x - c \rangle.$$

Now expand the inner product on both sides to recognize that this is equivalent to

$$\langle w, c' - c\rangle \leq \tfrac{1}{2}|c' - c|^2 \quad \text{for all} \quad c' \in C, \tag{1.2}$$

where we have relabeled $x - c$ as w.

Condition (1.2) can be related to a familiar one: the concept of supporting hyperplane. Indeed, if the quadratic term on the right side of (1.2) were replaced by zero, then (1.2) would assert that the hyperplane with normal direction w passing through the point c supports the set; i.e., all points in C lie on one side of the hyperplane. The presence of the quadratic term mollifies this; other points c' in C need not lie entirely on one side of the hyperplane, but they do so increasingly as c' approaches c. In fact, the supporting hyperplane is replaced by a supporting "hyperparabola."

The vector w is called a *perpendicular* to the set C at the point c. We can express the concept in alternative terms by means of the *distance function* $d_C(\cdot)$ associated with C:

$$d_C(x) := \inf_{c' \in C} |x - c'|.$$

Then w is a perpendicular to C at c provided that

$$d_C(c + w) = |w| > 0.$$

It is convenient to introduce a further term. A *proximal normal* to C at c is any nonnegative multiple of a perpendicular to C at c. Equivalently, in view of the defining relation (1.2), the vector ζ is a proximal normal to C at c if and only if for some positive scalar σ we have

$$\langle \zeta, c' - c\rangle \leq \sigma|c' - c|^2 \quad \text{for all} \quad c' \in C.$$

This is also equivalent to saying that some point of the form $c + t\zeta$ (for $t > 0$) has closest point $c \in C$.

We label $\Pi_C(c)$ the set of proximal normals to C at c. Note that it is not every point c on the boundary of C that admits a nontrivial (i.e., nonzero) proximal normal. The reader may verify that an example of this in $\mathbb{R}^2$ is obtained by taking $C = \{(x, y) : y + |x| \geq 0\}$, $c = (0, 0)$. Although nontrivial proximal normals need not exist at every boundary point, it is easy to see that any boundary point c is arbitrarily close to another $\tilde{c}$ that has that property (simply let x be a point near c but outside C, and take for $\tilde{c}$ the closest point in C to x). The density of such points helps motivate the following definition.

DEFINITION. Let $c \in C$. The prenormal cone[2] to C at c is the set

$$\hat{N}_C(c) := \{\lim_{i \to \infty} \zeta_i : \zeta_i \in \Pi_C(c_i),\ c_i \to c\}$$

and the normal cone to C is the set

$$N_C(c) := \operatorname{cl} \operatorname{co} \hat{N}_C(c).$$

[2]A cone is a set closed under positive scalar multiplication. The set $\hat{N}_C(c)$ is also known as the "limiting proximal normal cone." Its utility was first emphasized by Mordukhovich [M1980].

If c lies in the interior of C, then the ζ_i appearing in the definition of $\hat{N}_C(c)$ are necessarily zero for i sufficiently large, and so $\hat{N}_C(c)$ and $N_C(c)$ reduce to $\{0\}$. On the other hand, $\hat{N}_C(c)$ contains nonzero vectors whenever c lies on the boundary of C, thanks to the compactness of the unit ball in this finite-dimensional context.

Of course there exist other notions of normality, notably the classical ones when C is a smooth manifold, and the normal cone in the sense of convex analysis when C is convex. It is doubtful that any generalization of normality would be useful if it failed to reduce to one of these in the appropriate case; it can be shown that $N_C(c)$ passes this test.

The classical technique of linearization uses tangency for sets (and derivatives for functions). We obtain our tangents by applying *polarity*, a duality mechanism in the class of closed convex cones. Thus $T_C(c)$, the tangent cone to C at c, is defined by

$$T_C(c) := \{v : \langle v, \zeta \rangle \leq 0 \quad \text{for all} \quad \zeta \in N_C(c)\}. \tag{1.3}$$

Interchanging the symbols N_C and T_C in (1.3) leaves that relation valid and allows N_C to be computed if T_C is known. We could have made the tangent cone our fundamental definition, for it turns out that a direct characterization is possible, as follows: v belongs to $T_C(c)$ if and only if for every sequence c_i in C converging to c and every sequence t_i in $(0, \infty)$ converging to 0, there is a sequence v_i converging to v such that $c_i + t_i v_i$ belongs to C for all i.

There are many quantifiers in this last paragraph that can be changed or permuted to generate other notions. However, T_C enjoys certain properties that single it out as a good choice. One of these, apparent from (1.3) but not from the direct definition, is that $T_C(c)$ is a convex set.

We now have our basic geometrical constructs, and it is time to turn to analytical concepts for functions.

1.3. The subdifferential.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, extended-valued function. The *epigraph* of f is the set

$$\operatorname{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geq f(x)\}.$$

Let x be a point at which f is finite-valued. We define the generalized *subdifferential* of f at x, denoted $\partial f(x)$, as follows:

$$\partial f(x) := \{\zeta : (\zeta, -1) \in N_{\operatorname{epi} f}(x, f(x))\}. \tag{1.4}$$

We recognize here the basic idea that, classically, $\zeta = f'(x)$ is a vector such that $(\zeta, -1)$ is a (downward) normal to the graph of f. An element ζ of $\partial f(x)$ is termed a *subgradient*. The original nomenclature for the set $\partial f(x)$ [C1973] was the *generalized gradient*, and it is often referred to by this name.

It can be shown that if f is smooth (i.e., continuously differentiable), then $\partial f(x)$ reduces to $\{\nabla f(x)\}$, where ∇f is the gradient of f, and that if f is

convex, then $\partial f(x)$ coincides with the set of subgradients in the sense of convex analysis.[3]

A further useful concept that can be extended to the present setting is that of directional derivative; it is dual to the subdifferential in much the same way as the normal and tangent cones are dual to one another. We define $f^0(x;v)$, the generalized *directional derivative* of f at x in the direction v, as follows:

$$f^0(x;v) = \sup\{\langle v, \zeta\rangle : \zeta \in \partial f(x)\}. \tag{1.5}$$

This defines $f^0(x;\cdot)$ as the *support function* of the set $\partial f(x)$.

Just as N_C and ∂f are linked (when $C = \operatorname{epi} f$), it is possible to link T_C and f^0:

$$T_{\operatorname{epi} f}(x, f(x)) = \operatorname{epi} f^0(x;\cdot).$$

In view of the direct definition of the tangent cone mentioned above, this relationship induces a direct definition of f^0, following which $\partial f(x)$ can be defined via (1.5). This route has been mapped by Rockafellar. We shall not explore it here, other than to note for interest's sake the resulting eye-catching expression for $f^0(x;v)$ (B signifies the open unit ball in $\mathbb{R}^n$):

$$f^0(x;v) = \lim_{\epsilon \downarrow 0} \limsup_{\substack{y \to x, f(y) \to f(x) \\ t \downarrow 0}} \inf_{w \in v + \epsilon B} \frac{f(y+tw) - f(y)}{t}.$$

As we shall see, the possibility of treating extended-valued functions is especially useful in connection with implicit representation of constraints. Central to this technique is the *indicator function* of a given set C, denoted ψ_C and defined by

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that minimizing a real-valued function f over C is equivalent to (freely) minimizing the function $f + \psi_C$. For a point x in C, we have the formula $\partial \psi_C(x) = N_C(x)$. This provides the link to the necessary conditions that will be discussed presently.

An early example of the utility of the extended-valued formulation in a nonconvex setting appears in [C1975b], where the issue is that of necessary conditions for the problem of minimizing functionals involving the variational integral

$$\Lambda(x) := \int_0^T L(t.x(t), \dot{x}(t))\, dt.$$

We shall be studying such problems in the next chapter. The competing functions x are subject to various boundary constraints, and there may also be boundary functionals added to Λ in the objective functional. Examples include the

[3] A subgradient at x of the convex function f is a vector ζ such that $f(y) - f(x) \geq \langle y - x, \zeta\rangle$ for all y.

following:

$$(P_1) \qquad \begin{array}{l} \text{minimize } \Lambda(x) \\ \text{subject to } x(0) \in C_0, x(T) \in C_T, \end{array}$$

$$(P_2) \qquad \begin{array}{l} \text{minimize } f(x(T)) + \Lambda(x) \\ \text{subject to } x(0) = x_0 (x(T) \text{ is free}), \end{array}$$

$$(P_3) \qquad \begin{array}{l} \text{minimize } g(x(0), x(T)) + \Lambda(x) \\ \text{subject to } (x(0), x(T)) \in S. \end{array}$$

Here f and g would be locally Lipschitz functions and C_0, C_T, S closed sets. We do not discuss details at this time, other than to remark that the appropriate necessary conditions involve a function p that satisfies certain boundary conditions depending on the nature of the problem. These are known as *transversality conditions* and have the following form in each of the above cases:

$$(P_1) \qquad p(0) \in N_{C_0}(x(0)), \ -p(T) \in N_{C_T}(x(T)),$$

$$(P_2) \qquad -p(T) \in \partial f(x(T)),$$

$$(P_3) \qquad (p(0), -p(T)) \in \partial g(x(0), x(T)) + N_S(x(0), x(T)).$$

The device of using extended-valued functions allows us to group all three of these cases (and more) in a common setting. Indeed, consider the following problem (P):

$$(P) \qquad \text{minimize } \ell(x(0), x(T)) + \Lambda(x),$$

where now ℓ is permitted to be *extended-valued.* This subsumes $(P_1) - (P_3)$ above, and the appropriate choices of ℓ are:

$$(P_1) \qquad \ell(u, v) = \psi_{C_0}(u) + \psi_{C_T}(v),$$

$$(P_2) \qquad \ell(u, v) = \psi_{\{x_0\}}(u) + f(v),$$

$$(P_3) \qquad \ell(u, v) = g(u, v) + \psi_S(u, v).$$

In each case, the transversality condition for (P), which is

$$(p(0), -p(T)) \in \partial \ell(x(0), x(T)),$$

gives the appropriate one for the particular problem.

Despite the potential utility of extended-valued functions, we must acknowledge the trade-off to be made with the correspondingly higher level of technicality associated with their use. A very useful class of nonsmooth functions for which the level of technicality is moderate, and yet which accounts for a great majority of the applications of the theory, is that of (locally) Lipschitz functions.

A function f is *Lipschitz* near x if for some scalar K, for all points y and z in some neighborhood of x, we have

$$|f(y) - f(z)| \leq K|y - z|.$$

The smallest such K for which this inequality holds for all points in a given set is called the *Lipschitz rank* of f relative to the set. Lipschitz functions form a useful and natural subclass of (not necessarily differentiable) functions in a variety of contexts, and they automatically enjoy a number of important properties as regards nonsmooth calculus. For example, the subdifferential of a Lipschitz function is *upper semicontinuous*:

$$\zeta_i \in \partial f(x_i), \quad \zeta_i \to \zeta, \quad x_i \to x \Rightarrow \zeta \in \partial f(x).$$

This property has played a central role in many applications where perturbation and convergence techniques come into play. The question of when this property is valid for non-Lipschitz functions is a reasonably complex one that we do not pause to study.

Because locally Lipschitz functions are so useful, it is of interest to have criteria available to aid us in recognizing them. In this connection, let us define the *singular subdifferential* $\partial^\infty f(x)$ as the set of vectors ζ such that $(\zeta, 0)$ lies in $N_{\operatorname{epi} f}(x, f(x))$.[4] If we compare this to the definition of $\partial f(x)$ given by (1.4) (bearing in mind the geometrical relation in the classical case between $f'(x)$ and normal vectors to the graph of f), we are led to suspect that f is Lipschitz near x precisely when $N_{\operatorname{epi} f}(x, f(x))$ contains no "horizontal" elements, that is, when $\partial^\infty f(x)$ reduces to $\{0\}$. The following result confirms this and summarizes some further properties of locally Lipschitz functions.

PROPOSITION 1.1. *The following are equivalent:*

(a) *f is Lipschitz near x;*
(b) *$\partial f(x)$ is nonempty and bounded;*
(c) *$\partial^\infty f(x) = \{0\}$.*

When f is Lipschitz near x, we have

$$f^0(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t} = \max_{\zeta \in \partial f(x)} \langle \zeta, v \rangle. \tag{1.6}$$

In addition, for any set S of measure zero, we have

$$\partial f(x) = \operatorname{co}\{\lim_{i \to \infty} \nabla f(x_i) : f \text{ is differentiable at } x_i, x_i \notin S, x_i \to x\}. \tag{1.7}$$

Formula (1.7) implies that $\partial f(x)$ is intrinsic relative to zero-measure sets and clearly shows the "inevitability" of the subdifferential in a certain sense. If we seek to define a set-valued derivative $\tilde{\partial} f(x)$ of some kind that will always contain the derivative when it exists, and if we hold to the important upper semicontinuity property, then the set $\tilde{\partial} f(x)$ must always contain the one on the right side of (1.7) (before the convex hull is taken). If now we also wish to maintain the rich duality between subgradients and directional derivatives

[4] In [C1983], $\partial^\infty f(x)$ is referred to as the "asymptotic generalized gradient."

(or in geometrical terms, the polarity between normal and tangent cones), then convexity is essential and the set defined in (1.7) reveals itself as minimal.

Of course, we may explore the possibilities of giving up one or more of these properties, but there will inevitably be a price to pay. Whether we should pay it depends also on the breadth of the applications we wish to address. For example, to mention a context that has been frequently cited, if the only issue of interest is whether we have $0 \in \hat{\partial} f(x)$ whenever x minimizes f, then there are definitions of the generalized subgradient that give potentially finer results than ours. This is not surprising, since upper semicontinuity and duality do not play an important role in this limited context. Even in this case, however, the subdifferential takes account of local behavior in such a way that makes it the preferred choice for issues of stability, computing, and sensitivity.[5]

There is, however, a "presubdifferential" $\hat{\partial} f$, related to ∂f much as $\hat{N}_C$ is related to N_C, which has properties making it a useful tool for developing subdifferential calculus. We first introduce a functional version of the proximal normal concept. We say that ζ is a *proximal subgradient* (the term is due to Rockafellar [R1981]) of f at x, provided that for some $\sigma > 0$ we have, for all y in some neighborhood of x,

$$f(y) - f(x) + \sigma|y - x|^2 \geq \langle \zeta, y - x \rangle.$$

A bit of thought will confirm that this is equivalent to the condition that $(\zeta, -1)$ lies in $\Pi_{\text{epi}\, f}(x, f(x))$. We denote by $\partial^\pi f(x)$ the set of proximal subgradients of f at x. The *presubdifferential*[6] is defined to be the set

$$\hat{\partial} f(x) := \{ \lim_{i \to \infty} \zeta_i : \ \zeta_i \in \partial^\pi f(x_i),\ x_i \to x,\ f(x_i) \to f(x) \}, \tag{1.8}$$

whose elements are referred to as presubgradients. Of course the condition $f(x_i) \to f(x)$ is redundant when f is continuous at x. The proximal subgradient provides another route that leads to $\partial f(x)$ for general f. To describe how this is done, we also need the set of singular presubgradients, a set that plays a role similar to that of $\partial^\infty f(x)$:

$$\hat{\partial}^\infty f(x) := \{ \lim_{i \to \infty} t_i \zeta_i : \ \zeta_i \in \partial^\pi f(x_i),\ x_i \to x,\ f(x_i) \to f(x),\ t_i \downarrow 0 \}.$$

Rockafellar has shown [R1985] that in calculating $\hat{N}_{\text{epi}\, f}(x, f(x))$, the proximal normals $[\zeta_i, -\epsilon_i]$ whose limits generate that cone may be restricted to those having $\epsilon_i > 0$. We can deduce readily from this the following characterization:

$$\hat{N}_{\text{epi}\, f}(x, f(x)) = \{(t\zeta, -t) : \ \zeta \in \hat{\partial} f(x), t > 0\} \cup \hat{\partial}^\infty f(x) \times \{0\}.$$

This evidently implies that the analogue of (1.4) holds for $\hat{\partial} f$ and $\hat{N}_{\text{epi}\, f}$, a relationship that could have served, therefore, as the definition of $\hat{\partial} f(x)$ (which was the approach adopted by Mordukhovich).

The following proposition summarizes some facts about presubgradients.

[5] A thorough discussion of these and related points appears in [R1987]. See also [P1987] concerning the numerical use of subgradients.

[6] This is also known as the set of limiting proximal subgradients.

PROPOSITION 1.2. *The function f is Lipschitz near x if and only if $\hat{\partial}^\infty f(x) = \{0\}$. In that case we have $\partial f(x) = \operatorname{co} \hat{\partial} f(x)$; in general we have*

$$\partial f(x) = \operatorname{cl} \operatorname{co} \{\hat{\partial} f(x) + \hat{\partial}^\infty f(x)\}.$$

Formulas (1.6) and (1.7), which express f^0 and ∂f directly in the Lipschitz case, have no equivalent in terms of $\hat{\partial} f$. In contrast to $\partial f(x)$, the presubdifferential $\hat{\partial} f(x)$ is not stable under "excluding sets of measure zero." To be more precise, even when S is countable, adding a condition $x_i \notin S$ to the definition (1.8) leads to a set that depends in general on S.

In the past, (1.6) and (1.7) have been the starting points of the entire theory, which has always treated non-Lipschitz functions (although only after the Lipschitz case and then the geometrical theory were developed). This bootstrapping approach can be carried out with the help of the distance function $d_C(\cdot)$, which is globally Lipschitz of rank 1. It can be shown that d_C generates the normal and tangent cones as follows:

$$T_C(c) = \{v : d_C^0(c; v) = 0\}, \tag{1.9a}$$

$$N_C(c) = \operatorname{cl} \left\{ \bigcup_{\lambda \geq 0} \lambda \partial d_C(c) \right\}. \tag{1.9b}$$

These characterizations demonstrate that defining ∂f and f^0 for Lipschitz functions suffices to generate, for example, N_C, whereupon (1.4) can be used to extend the definition of ∂f to the non-Lipschitz case.

The distance function has played a very useful role in constrained optimization, in part due to a technique called *exact penalization*. Here is a specific simple instance, in the context of minimizing a Lipschitz function f on a closed set C.

PROPOSITION 1.3. *Let x minimize f over C, and let r exceed the Lipschitz rank of f relative to C. Then x is a local unconstrained minimizer of the function $f + r d_C$.*

Proof. There exists ϵ such that f has Lipschitz rank not exceeding r on the open ball $B(\epsilon, x)$. Now choose any y in the ball $B(\epsilon/2, x)$, and let z be a point in C closest to y. Since x itself lies in C, it follows that $|z - y| \leq \epsilon/2$, and hence that z belongs to $B(\epsilon, x)$. We calculate

$$\begin{aligned} f(x) &\leq f(z) \qquad (x \text{ minimizes } f \text{ over } C) \\ &\leq f(y) + r|y - z| \qquad (\text{the Lipschitz condition}) \\ &= f(y) + r d_C(y) \qquad (\text{definition of } z). \end{aligned}$$

This completes the proof, since y is any point in $B(\epsilon/2, x)$. □

When a function g has a local minimum at a point x, it follows that 0 belongs to $\partial g(x)$ (or to $\hat{\partial} g(x)$). A corollary, therefore, of the proposition is the following inclusion:

$$0 \in \partial\{f + r d_C\}(x).$$

Can such a relation be decomposed; i.e., expressed in terms of ∂f and ∂d_C ? We now turn to the study of such questions.

1.4. Some more calculus.

Nonsmooth calculus has been developed in varying degrees of generality. The price to pay for the greatest generality is heavy in terms of technicality; therefore, we shall begin with a discussion of Lipschitz functions, a class that comprises the principal applications and one for which results can be stated in simple terms.

The first class of results is one that consists of extending to the nonsmooth setting results known in the smooth case. For example, a *mean value theorem* due to Lebourg states that given x and y, there exists a point z in the open line segment joining x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x \rangle.$$

This clearly reduces to the corresponding classical result when f is smooth.

Consider next the formula $(f+g)' = f'+g'$, extensions of which we shall discuss in some detail in this section. Perhaps we would expect to have in general

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

However, this cannot be the case. Let f be a function whose generalized gradient at x is not a singleton. Then set $g = -f$. The last formula would give (using the facts that $\partial(-f) = -\partial f$ and that the subdifferential of a constant function is $\{0\}$)

$$\{0\} = \partial f(x) - \partial f(x),$$

which cannot be, failing a more elaborate definition of set subtraction. What turns out to hold in general is an inclusion:

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x). \tag{1.10}$$

How do we prove a formula such as (1.10)? A strength of our theory is that it involves both analytical and geometrical constructs, with all the components well linked to one another. Thus, in specific instances, we can have recourse to the point of view most suited to the problem at hand.

To prove (1.10) in a general non-Lipschitz setting, for example (which necessitates certain additional technical hypotheses), the best approach is geometric: we can argue via proximal normals to the various epigraphs. The nature of such an argument will be explored later. But in the present Lipschitz setting, an immediate proof is at hand via generalized directional derivatives, as we proceed to show.

Recall first the notion of the support function σ_C of a set C:

$$\sigma_C(x) := \sup_{c \in C} \langle x, c \rangle.$$

If C and D are closed convex sets, then the following equivalence is a consequence of the Hahn-Banach theorem:

$$C \subseteq D \iff \sigma_C(x) \leq \sigma_D(x) \quad \text{for all } x.$$

Now, in view of Proposition 1.1, the support functions of the left and right sides of (1.10) are, respectively, $(f + g)^0(x; \cdot)$ and $f^0(x; \cdot) + g^0(x; \cdot)$. Thus it suffices to prove the inequality (for any v)

$$(f + g)^0(x; v) \leq f^0(x; v) + g^0(x; v).$$

The left side of this last relation is given by

$$\limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tv)+g(y+tv)-f(y)-g(y)}{t},$$

which is evidently no greater than

$$\limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tv)-f(y)}{t} + \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{g(y+tv)-g(y)}{t} = f^0(x;v)+g^0(x;v).$$

And so (1.10) is proven.

It is natural to seek conditions under which equality holds in (1.10) and in numerous other calculus rules that we have not discussed. Some special cases of this type can be identified: when f and g are convex or when at least one of them is C^1 (in the latter case, the characterization (1.7) leads to an immediate proof). A more systematic study of the issue leads us to define a useful class of functions as follows: f is said to be (subdifferentially) *regular* at x provided that the usual one-sided directional derivative function $f'(x;\cdot)$ exists and coincides with $f^0(x;\cdot)$. Here $f'(x;v)$ designates

$$\lim_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

The class of regular functions enjoys certain closure properties and includes those functions that are convex and those that are C^1. A variety of subdifferential formulas that are inclusions in general specialize to equalities in the presence of regularity. For example (exercise) equality holds in (1.10) if f and g are regular at x (and in that case $f+g$ is regular at x).

There are formulas of subdifferential calculus that, in contrast to the ones discussed above, have no counterpart in classical calculus. Consider, for example, a function f expressible as the pointwise maximum of a finite collection of smooth functions:

$$f(x) := \max_{1\le i\le m} f_i(x).$$

It can be shown that f is regular, but such a function is virtually never differentiable everywhere, no matter how smooth the individual functions f_i. A formula for ∂f can be proven by means similar to those used to prove the sum formula. We get

$$\partial f(x) = \operatorname{co}\{\nabla f_i(x) : i \in I(x)\},$$

where $I(x)$ is the set of indices at which the maximum defining $f(x)$ is attained.

Nonsmooth analysis also includes a geometrical calculus that, for example, finds hypotheses under which we may assert the formula

$$N_{C_1\cap C_2}(x) \subseteq N_{C_1}(x) + N_{C_2}(x). \tag{1.11}$$

In fact, the general geometrical and the general non-Lipschitz calculus are essentially one and the same. Consider, for example, the evident relationship among indicator functions

$$\psi_{C_1\cap C_2} = \psi_{C_1}(x) + \psi_{C_2}(x),$$

which shows that (1.11) is formally equivalent to the sum formula (1.10) applied to indicator functions. A further ingredient of the general theory involves extending the notion of regularity to non-Lipschitz functions and to sets [C1983].

We pursue our brief look at subdifferential calculus with a result that extends (1.10) to non-Lipschitz settings. Besides imparting the flavor of these arguments, this extension will serve to illustrate "proximal precalculus," which is revealing itself as perhaps the simplest and most powerful method available to develop general subdifferential calculus.

It may seem unlikely that proximal subgradients would admit an interesting calculus, since $\partial^\pi f(x)$ would seem potentially empty for many x. However, we shall see for sums that, modulo small perturbations, certain facts hold in surprising generality. The approach taken to prove the next result[7] is quite in keeping with the spirit of this book: we analyze proximally a certain value function associated with the perturbations. The case $\epsilon = 0$ of the following result, false in general, would be the exact sum formula for the subdifferential ∂^π.

PROPOSITION 1.4. *Let f and g be extended-valued lower semicontinuous functions finite at x^*, and let ζ belong to $\partial^\pi(f+g)(x^*)$. Then for any $\epsilon > 0$ there exist y^* and z^* both belonging to $x^* + \epsilon B$, such that $f(y^*) \in f(x^*) + \epsilon B$, $g(z^*) \in g(x^*) + \epsilon B$, and such that ζ lies in $\partial^\pi f(y^*) + \partial^\pi g(z^*) + \epsilon B$.*

Proof. By definition, there is a compact neighborhood Ω of x^* such that the function

$$x \longrightarrow f(x) + g(x) - \langle \zeta, x\rangle + \sigma|x - x^*|^2$$

has a minimum over $x \in \Omega$ at $x = x^*$. By increasing σ if necessary, we can assume that x^* is the unique minimum. Let us define $V(\alpha,\beta)$ to be the minimum over Ω of the perturbed function

$$x \longrightarrow f(x+\alpha) + g(x) - \langle \zeta, x\rangle + \sigma|x + \beta - x^*|^2.$$

It follows readily that V is lower semicontinuous. Further, for each (α,β), there is at least one solution $x_{\alpha,\beta}$ of this minimization problem. Because x^* is the only solution for $(\alpha,\beta) = (0,0)$, it is an easy exercise in compactness and lower semicontinuity to show that for every $\Delta > 0$ there exists $\delta > 0$ such that the following implication is valid:

$$\left.\begin{array}{r} |(\alpha,\beta)| < \delta \\ |V(\alpha,\beta) - V(0,0)| < \delta \end{array}\right\} \implies \left\{\begin{array}{l} |x_{\alpha,\beta} - x^*| < \Delta \\ |f(x_{\alpha,\beta}+\alpha) - f(x^*)| < \Delta \\ |g(x_{\alpha,\beta}) - g(x^*)| < \Delta. \end{array}\right.$$

Now let δ correspond in this way to some $\Delta < \epsilon/2$, with $\delta < \epsilon/2$, and $2\sigma(\delta+\Delta) < \epsilon$, and let (α,β) be a point as above at which V admits a proximal normal (γ,ψ). Thus for some $\tilde\sigma > 0$, the following function of (α',β') assumes a local minimum at $(\alpha',\beta') = (\alpha,\beta)$:

$$V(\alpha',\beta') - \langle \gamma, \alpha'\rangle - \langle \psi, \beta'\rangle + \tilde\sigma\{|\alpha' - \alpha|^2 + |\beta' - \beta|^2\}. \tag{1.12}$$

[7] We adapt the argument from [CR1989], [RA1987]. This "fuzzy calculus" is also developed in [I1984].

Now let z^* be a solution to the problem defining $V(\alpha,\beta)$, i.e., $z^* = x_{\alpha,\beta}$. We make two observations. First, for any (α',β') near (α,β) and for any z in Ω, the term $V(\alpha',\beta')$ is majorized by

$$f(z+\alpha') + g(z) - \langle \zeta, z\rangle + \sigma|z+\beta'-x^*|^2 \tag{1.13}$$

by definition of V. Thus if this expression replaces $V(\alpha',\beta')$ in (1.12), the latter can only increase. Second, if we take $z = z^*$, $(\alpha',\beta') = (\alpha,\beta)$, then the value of (1.13) becomes $V(\alpha,\beta)$. Together, these observations imply that for any x in Ω and for any (α',β') near (α,β), we have $h(z,\alpha',\beta') \geq h(z^*,\alpha,\beta)$, where h is given by

$$\begin{aligned} f(z+\alpha') + g(z) - \langle \zeta, z\rangle + \sigma|z+\beta'-x^*|^2 \\ - \langle \gamma, \alpha'\rangle - \langle \psi, \beta'\rangle + \tilde{\sigma}\{|\alpha'-\alpha|^2 + |\beta'-\beta|^2\}. \end{aligned}$$

To make this fact more useful, let us change notation by setting $z+\alpha' =: y$, $z+\beta' =: u$, $z^*+\alpha =: y^*$, $z^*+\beta =: u^*$. Then our conclusion regarding h translates into the statement that for any (z,y,u) near (z^*,y^*,u^*) (which ensures that α' is near α, β' near β), the function

$$\begin{aligned} f(y) + g(z) - \langle \zeta, z\rangle + \sigma|u-x^*|^2 - \langle \gamma, y\rangle + \langle \gamma, z\rangle \\ - \langle \psi, u\rangle + \langle \psi, z\rangle + \tilde{\sigma}\{|y-z-y^*+z^*|^2 + |u-z-u^*+z^*|^2\} \end{aligned}$$

attains a value no less than that at $(z,y,u) = (z^*,y^*,u^*)$. The last term in this expression is bounded above by

$$2\tilde{\sigma}\{|y-y^*|^2 + 2|z-z^*|^2 + |u-u^*|^2\},$$

and it follows that the following function of (z,y,u) achieves a local minimum at (z^*,y^*,u^*):

$$\begin{aligned} f(y) + g(z) - \langle \zeta, z\rangle + \sigma|u-x^*|^2 - \langle \gamma, y\rangle + \langle \gamma, z\rangle \\ - \langle \psi, u\rangle + \langle \psi, z\rangle + 2\tilde{\sigma}\{|y-y^*|^2 + 2|z-z^*|^2 + |u-u^*|^2\}. \end{aligned}$$

If we fix $y = y^*$, $u = u^*$ in this function, we then reduce to a local minimum with respect to the z variable. The nature of the function involved implies by definition that we have

$$\zeta - \gamma - \psi \in \partial^\pi g(z^*).$$

Similarly, looking at the y variable yields $\gamma \in \partial^\pi f(y^*)$. Finally, focusing on the u variable, in which the dependence is smooth, we note that the vanishing of the derivative at $u = u^*$ gives $\psi = 2\sigma(u^* - x^*)$. We have

$$|u^* - x^*| = |z^* + \beta - x^*| \leq |z^* - x^*| + |\beta| < \Delta + \delta,$$

so that our choice of δ and Δ assures that $\psi \in \epsilon B$. Note now that $\zeta = \gamma + (\zeta - \gamma - \psi) + \psi$, which expresses ζ exactly in the required form. □

Once we have Proposition 1.4, it is an easy matter to derive the sum formula for the limiting subdifferentials. Formula (a) in the following proposition is due to Mordukhovic [M1984].

PROPOSITION 1.5. *Let f and g be extended-valued lower semicontinuous functions finite at x, with $\hat{\partial}^\infty f(x) \cap -\hat{\partial}^\infty g(x) = \{0\}$. Then we have*

(a) $\hat{\partial}(f+g)(x) \subseteq \hat{\partial}f(x) + \hat{\partial}g(x)$,
(b) $\hat{\partial}^\infty(f+g)(x) \subseteq \hat{\partial}^\infty f(x) + \hat{\partial}^\infty g(x)$,
(c) $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$.

Proof. Let ζ belong to $\hat{\partial}(f+g)(x)$. Then by definition we have $\zeta = \lim \zeta_i$, where $\zeta_i \in \partial^\pi(f+g)(x_i)$, $x_i \to x$. We apply Proposition 1.4 to write ζ_i as $\gamma_i + \psi_i + \eta_i$, where $\gamma_i \in \partial^\pi f(y_i^*)$, $\psi_i \in \partial^\pi g(z_i^*)$, $\eta_i \to 0$, and where y_i^*, z_i^* converge to x; $f(y_i^*)$ to $f(x)$; and $g(z_i^*)$ to $g(x)$. Suppose first that the sequence γ_i is bounded. Then ψ_i is also bounded necessarily, and we may suppose that ψ_i and γ_i converge to limits ψ and γ, respectively, with $\zeta = \gamma + \psi$. But then $\gamma \in \hat{\partial}f(x)$ and $\psi \in \hat{\partial}g(x)$ by definition, and so $\zeta \in \hat{\partial}f(x) + \hat{\partial}g(x)$ as required. The argument is the same when the sequence ψ_i is assumed bounded, so the remaining case is when both $|\gamma_i|$ and $|\psi_i|$ go to $+\infty$. We may assume that $\gamma_i/|\gamma_i|$ and $\psi_i/|\psi_i|$ converge; the limits are unit vectors γ_0 and ψ_0 in $\hat{\partial}^\infty f(x)$ and $\hat{\partial}^\infty g(x)$, respectively.

We claim that $\lim_{i\to\infty} |\gamma_i|/|\psi_i| = 1$. Indeed, we have

$$\begin{aligned}\lim_{i\to\infty} \frac{|\gamma_i|}{|\psi_i|} &\le \lim_{i\to\infty} \frac{|\gamma_i - (\gamma_i + \psi_i)| + |\gamma_i + \psi_i|}{|\psi_i|} \\ &= 1 + \lim_{i\to\infty} \frac{|\zeta_i - \eta_i|}{|\psi_i|} = 1;\end{aligned}$$

by symmetry the first limit must equal 1. If we now divide across by $|\gamma_i||\psi_i|$ in the identity

$$\langle \gamma_i, \psi_i \rangle = \tfrac{1}{2}\{|\gamma_i + \psi_i|^2 - |\gamma_i|^2 - |\psi_i|^2\}$$

and then pass to the limit, we obtain $\langle \gamma_0, \psi_0 \rangle = -1$. This shows that $-\psi_0 = \gamma_0$, so that $\hat{\partial}^\infty f(x) \cap -\hat{\partial}^\infty g(x) \neq \{0\}$, contrary to assumption. This proves (a).

To prove (b), we examine any point ζ of $\hat{\partial}^\infty(f+g)(x)$, which can be written $\lim t_i(\gamma_i + \psi_i + \eta_i)$ much as in the proof of (a), where the sequence t_i decreases to 0. The case in which either γ_i or ψ_i is bounded is trivial, so we may suppose that both go to infinity in norm. If now either $t_i\gamma_i$ or $t_i\psi_i$ is bounded, then once again the result follows. When $|t_i\gamma_i|$ and $|t_i\psi_i|$ both go to infinity, the argument used to end the proof of (a), applied to $t_i\gamma_i$ and $t_i\psi_i$, again completes the proof.

The formula (c) follows, with the help of Proposition 1.2:

$$\begin{aligned}\partial(f+g)(x) &= \operatorname{cl}\operatorname{co}\{\hat{\partial}(f+g)(x) + \hat{\partial}^\infty(f+g)(x)\} \\ &\subseteq \operatorname{cl}\operatorname{co}\{\hat{\partial}f(x) + \hat{\partial}g(x) + \hat{\partial}^\infty f(x) + \hat{\partial}^\infty g(x)\} \\ &\subseteq \operatorname{cl}\operatorname{co}\{\hat{\partial}f(x) + \hat{\partial}^\infty f(x)\} + \operatorname{cl}\operatorname{co}\{\hat{\partial}g(x) + \hat{\partial}^\infty g(x)\} \\ &= \partial f(x) + \partial g(x)\end{aligned}$$

□

The reader is invited to prove the following consequence.[8] (Hint: $\hat{\partial}^\infty \psi_C(x) = \hat{N}_C(x)$.)

[8]The first results in this vein are due to Aubin [AE1984] and Rockafellar [R1980].

COROLLARY. *If $x \in C_1 \cap C_2$, where C_1 and C_2 are closed sets satisfying $\hat{N}_{C_1}(x) \cap -\hat{N}_{C_2}(x) = \{0\}$, then* (1.11) holds.

Remark. The hypothesis $\hat{\partial}^\infty f(x) \cap -\hat{\partial}^\infty g(x) = \{0\}$ is automatically satisfied if either f or g is Lipschitz near x, so the proposition clearly extends (1.10), which we proved earlier for Lipschitz functions. In fact, since Proposition 1.5(a) is in terms of $\hat{\partial}$, an object smaller than ∂ in general, it is a finer estimate leading in certain contexts to potentially more precise results. As a matter of practicality, we generally prefer to work in terms of ∂f, the generalized gradient, in subsequent applications. This is due in part to its more appealing calculus. For example, in the Lipschitz case we have noted in (1.7) that ∂f is invariant to "excluding sets of measure 0", and we also have the formula $\partial(-f) = -\partial f$. Both these facts fail for $\hat{\partial} f$. Furthermore, the duality between ∂f and f^0 embodied in (1.5), together with (1.7), are perhaps the facts most exploited in the host of estimates implicit or explicit in later arguments, and like the polarity between N_C and T_C, are unavailable in limiting proximal terms. Another fact leading us to favor ∂f in later chapters is the use of limiting arguments involving inclusions such as $v_i(t) \in \partial f(x_i(t))$. When the convergence of v_i is weak, which is typically the case, the convexity of ∂f is essential to deduce the limiting inclusion. Having said this, we note that the presubdifferential $\hat{\partial} f$ has emerged as a very useful tool in developing generalized calculus, as we have seen, as well as in certain issues related to finite-dimensional necessary conditions, as we shall see.

1.5. Generalized Jacobians.

A number of situations (for example, the sum and the pointwise maximum just considered) can be viewed as special cases of the following:

$$f(x) = g(F(x)),$$

where F maps $\mathbb{R}^n$ to $\mathbb{R}^m$ and g maps $\mathbb{R}^m$ to $\mathbb{R}$. It is natural, therefore, to seek chain rules for nonsmooth *vector-valued* functions F, rules that would necessitate an extension of the theory of subdifferentials to vector functions.

One approach that comes to mind is to define

$$\partial F(x) = \partial f_1(x) \times \partial f_2(x) \cdots \times \partial f_m(x),$$

where $F = [f_1, f_2, \cdots, f_m]$. But this turns out to be too coarse an object. When F is locally Lipschitz, a smaller and better definition for $\partial F(x)$, the *generalized Jacobian* of F at x, may be based on the property (1.7):

$$\partial F(x) = \operatorname{co}\left\{\lim_{i\to\infty} JF(x_i) : F \text{ is differentiable at } x_i, x_i \to x\right\}. \tag{1.14}$$

Here JF denotes the customary $m \times n$ Jacobian matrix of partial derivatives.[9] Using this definition, we can prove useful chain rules that include the special

[9]It follows from the work of Warga [W1981] on derivate containers that $\partial F(x)$ is indifferent to the exclusion of x_i in (14) from an arbitrary set of measure 0, preanswering a question in [C1983].

cases we have mentioned, and that assert, in the context mentioned above, the relationship

$$\partial f(x) \subset \text{co}\{\partial g(F(x))\partial F(x)\}.$$

Again, proximal arguments can be used to obtain more refined chain rules much as they were employed to prove the sum formula in the preceding section; we shall not go into detail.

As an illustration of the use of generalized Jacobians, let us consider the issue of solving for x the equation

$$F(x) = y$$

for (x, y) near a point (x_0, y_0) such that $F(x_0) = y_0$. (Here F maps $\mathbb{R}^n$ to $\mathbb{R}^n$.) The classical inverse function theorem asserts that when F is C^1 and $JF(x_0)$ is invertible (nonsingular), then F^{-1} exists locally around y_0 as a C^1 function. It has been proven [C1976a] that when F is merely Lipschitz, when $\partial F(x_0)$ is nonsingular (i.e., every matrix in $\partial F(x_0)$ is nonsingular), then F^{-1} exists locally as a Lipschitz function. It is easy to give examples showing that this conclusion fails if we merely assume that JF is nonsingular when it exists, or if "co" is deleted in the definition (1.14) of $\partial F(x_0)$.

1.6. The infinite-dimensional case.

We consider now the situation in which x belongs to a Banach space X instead of to $\mathbb{R}^n$. Much of what has been done so far can be extended to this more general setting. In previous sections, we have alluded to several alternative paths to the theory. Here, for example, are two that can be traced in an infinite-dimensional setting.

(a) Define $T_C(c)$ by the direct definition given in §1.2, then $N_C(c)$ by the polarity relationship (see (1.3)), and proceed to define $\partial f(x)$ via (1.4) and, $f^0(x; v)$ via (1.5).

(b) Define $f^0(x; v)$ for Lipschitz functions f via (1.6), and then $\partial f(x)$ by (1.5). Use these with $f = d_C$ to arrive at N_C and T_C via (1.9a) and (1.9b), and then define ∂f, f^0 for non-Lipschitz functions by (1.4) and (1.5).

(The second route was chosen in [C1983].) Yet another possibility was adopted in [C1973], where (1.7) was the starting point. This path, based on Rademacher's theorem, is problematical in infinite dimensions.[10]

A long-standing problem has been to determine whether the line we have adopted in this chapter (namely, to use the proximal normal as the fundamental concept) could be adapted to an infinite-dimensional setting.

A positive answer to this question has now been provided by Borwein and Strojwas [BS1986] for certain Banach spaces, in particular for Hilbert spaces.[11] We limit ourselves here to this latter case. Borwein and Strojwas proved that the normal cone $N_C(c)$ (which, as we have said, can be defined in various ways

[10] A version of (1.7) has been obtained for certain separable spaces [HUT1980]. Preiss [PR1989] has recently announced its validity in any Banach space whose norm is differentiable.

[11] The Hilbert space case is very clearly expounded by Loewen in [L1987a].

in infinite dimensions without reference to proximal normals) continues in fact to be generated by proximal normals as described in §1.2:

$$(1.15)\quad N_C(c) = \operatorname{cl}\operatorname{co}\{w - \lim_{i\to\infty} \zeta_i : \zeta_i \text{ is a proximal normal to } C \text{ at } c_i, c_i \to c\}$$

(the convergence of c_i to c is strong, and $w-\lim$ denotes limit in the weak topology). This then extends the proximal normal formula of [C1973] to infinite dimensions and allows once again the choice of proximal normals as the basic concept. We shall see in later chapters the utility of this formula, in both its finite- and infinite-dimensional versions, in connection with the subdifferential analysis of value functions.

Although the general framework and calculus that we have surveyed in the preceding sections can be and has been extended to infinite dimensions, some care must be exercised due to pathologies that become possible. For example, in contrast to the finite-dimensional case, a point x on the boundary of a closed set C may be such that $N_C(x)$ reduces to $\{0\}$. Or in functional terms, in contrast to the assertion of Proposition 1.1, a function f may have $\partial^\infty f(x) = \{0\}$ and yet fail to be Lipschitz near x. And of course it becomes a more delicate matter to develop the calculus. However, the limiting proximal approach described in §1.4 has been successfully extended to infinite dimensions. For this and related results see [BFG1987], [BG1987], [I1986], [I1989], [L1988], [T1983], [T1986], [WB1987].

1.7. Value functions.

Suppose that an optimization problem (P) is given in the following abstract form:

$$(P)\qquad \begin{array}{l}\text{minimize } f(x)\\ \text{subject to } x \in X.\end{array}$$

Here, X could be a subset of an infinite-dimensional space. It often happens that (P) lends itself naturally to parametric perturbation, so that (P) is imbedded in a family of optimization problems (P_α) indexed by a parameter α:

$$(P_\alpha)\qquad \begin{array}{l}\text{minimize } f(x,\alpha)\\ \text{subject to } x \in X(\alpha).\end{array}$$

The value (i.e., the minimum) of the problem (P_α) is denoted $V(\alpha)$, and $V(\cdot)$ is called the value function. Even when our attention continues to center around the original problem (P), the function V is quite often of great interest in its analysis. We proceed to examine a few typical properties of V that have a bearing on (P).

(i) $V(\alpha)$ *finite for* α *near* 0. The set $X(\alpha)$ may well be empty for certain values of α, in which case $V(\alpha)$ is assigned the value $+\infty$, by the usual convention regarding the infimum over the empty set. The very finiteness of V is therefore of interest, and the stated property (i), implying as it does that $X(\alpha)$ is nonempty for α near 0, is interpretable in specific cases as a *solvability* or *controllability* result.

(ii) *V continuous or Lipschitz near* 0. Depending on the specific nature of the problem, the continuity of V at $\alpha = 0$ may be thought of as a necessary condition for a problem to be well posed, in other words, to exhibit a reasonably behaved dependence on initial conditions. A Lipschitz condition such as

$$|V(\alpha) - V(0)| \leq K|\alpha|$$

is a *stability* property of interest in, for example, error estimates, for certain algorithms that solve exactly not (P) $(= (P_0))$, but instead (P_α) for some α whose size can be estimated.

(iii) *V is differentiable at* 0. If $\nabla V(0)$ exists and can be calculated, it becomes possible to calculate the asymptotic effect of small parametric perturbations. It may happen that (P_0) is relatively simple to solve (analytically or numerically), whereas (P_α) is not. For example, $\alpha \neq 0$ could represent a nonautonomous perturbation of an originally autonomous optimal control problem, or a stochastic element of the problem that is absent for $\alpha = 0$. In such cases, an expansion for $V(\alpha)$ of the form $V(0) + \langle \alpha, \nabla V(0) \rangle + o(\alpha)$ can be useful as a check on the correctness and convergence of numerical procedures for calculating $V(\alpha)$.

(iv) *V has one-sided directional derivatives at* 0. This is a weaker property than (iii) and is of interest for similar reasons.

(v) *V satisfies a functional relation such as a differential equation.* Such a property would offer the hope of solving (P) (and perhaps (P_α)) by the indirect method of solving the relation in question. The canonical example here is the Hamilton–Jacobi equation, which we will discuss in Chapter 3.

It is apparent from the list that differential properties of V are of inherent interest, so a natural question is whether V is differentiable. To shed some light on this, let us look at a special case of the general situation, in which f is unperturbed and $X(\alpha)$ is the set of x (in some space) satisfying $h(x) = \alpha$, where h is a given function embodying the equality constraint of this mathematical programming problem. Thus we have

$$V(\alpha) = \min \{f(x) : h(x) = \alpha\}.$$

Assuming for simplicity that h is real-valued, we can study V by looking at the *image set* for this problem, the set

$$\{(h(x), f(x)) : x \text{ in the space in question}\}.$$

Note that this set would lie in $\mathbb{R}^2$ (say) even if the x space is infinite-dimensional. In FIG. 1 an example of a (smooth) image set is given. We shall study the behavior of V near a few judiciously chosen points. The value of $V(\alpha)$ corresponds to the ordinate of the lowest point lying in both the image set and the vertical line $h = \alpha$. The local nature of V is indicated by the parts of the image set curve that are emphasized.

At α_1 and nearby to the right, V is finite. But for any $\alpha < \alpha_1$, $V(\alpha) = +\infty$, since $h(x) = \alpha$ admits no solutions. Near α_2, V is well behaved (finite and

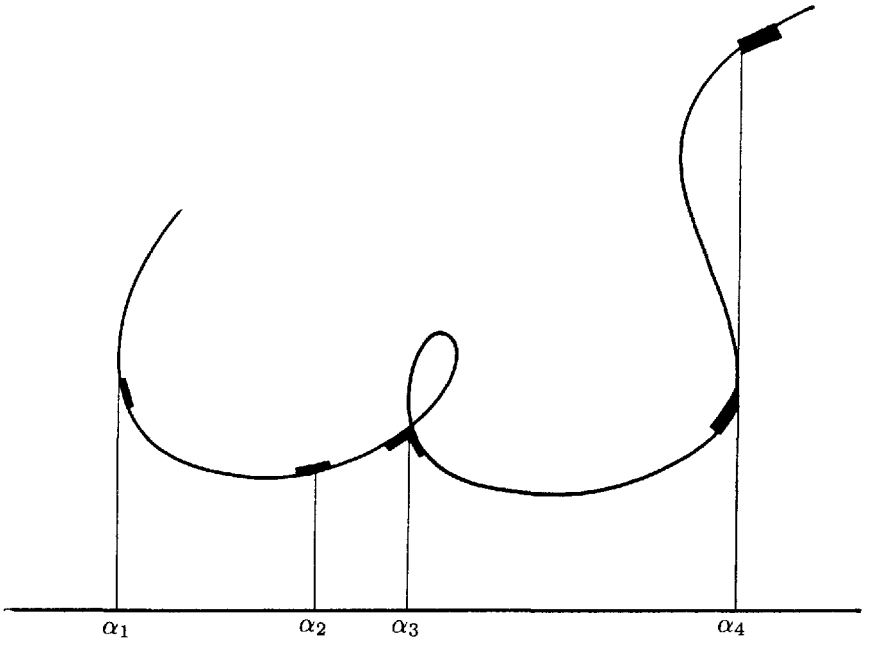

FIG. 1

smooth). Near α_3, V is continuous, but it has a "corner" (nondifferentiable point) at α_3 itself. Finally, V has a discontinuity at α_4 while being finite locally.

It follows from these observations that in studying differential properties of V, we must either find and impose a priori conditions that will imply that V is smooth at a given point, or else carry out an analysis that confronts the nondifferentiability of V. We expect that the second of these approaches will be broader in scope and will subsume the first as a special case.

1.8. The subdifferential of a value function.

We shall proceed now to illustrate the use of nonsmooth analysis to analyze the differential properties of value functions. We consider the mathematical programming problem (P) defined as follows:

$$(P) \qquad \begin{array}{l} \text{minimize } f(x) \\ \text{subject to } h(x) = 0, x \in C. \end{array}$$

Here C is a given subset of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ are given locally Lipschitz functions. We suppose the sets $\{x : f(x) \leq r, |h(x)| \leq 1, x \in C\}$ to be compact for all r.

The hypotheses of this fairly general problem do not include differentiability of the data, and the presence of the "abstract constraint" $x \in C$ is nonstandard. We wish to show how easily our methods handle such phenomena and to remark that they also apply to explicit inequality constraints $g(x) \leq 0$. The goal is not merely to generalize known results to nonsmooth data, but to show how proximal analysis of the value function leads to a better and simpler approach that yields more information even in the presence of smooth data. In fact, the method, which was introduced for the first time in [C1983], is indifferent to this factor.

We imbed (P) in a family (P_α) of similar problems, where $\alpha \in \mathbb{R}^m$ represents a perturbation of the equality constraint

$$(P_\alpha) \qquad \text{minimize } f(x) : \ h(x) + \alpha = 0, x \in C.$$

We let $V(\alpha)$ be the minimum in problem (P_α). Note that our assumptions imply that this minimum is attained finitely, unless the "feasible set" for (P_α) (namely, the set of points x in C satisfying $h(x) + \alpha = 0$) is empty, in which case $V(\alpha) = +\infty$. We assume that at least for $\alpha = 0$, this feasible set is nonempty. Our principal goal is to study $\partial V(0)$; the interest in doing so will become apparent subsequently.

There is a number K such that in a neighborhood of C (to be specific, let us say on the set $\{x : d_C(x) \leq 1\}$), the function $[f, h]$ is Lipschitz of rank K. Let r be any number exceeding K, and let us label as a *multiplier* corresponding to x any vector $\lambda \in \mathbb{R}^m$ such that

$$0 \in \partial_x L(x, \lambda),$$

where L, the *Lagrangian*, is defined via

$$L(x, \lambda) := f(x) + \langle \lambda, h(x) \rangle + r|(1, \lambda)| d_C(x).$$

When f and h are smooth and $C = \mathbb{R}^n$, we recognize in the multiplier relation above the classical Lagrange multiplier rule:

$$0 = f'(x) + \lambda h'(x).$$

The way in which the set C appears in the Lagrangian, through the term $r|(1, \lambda)| d_C(x)$, is an innovation that leads to sharper results than would, for instance, the possibly more natural-looking choice of ψ_C, the indicator function of C.

We denote by $M(x)$ the set of multipliers corresponding to x and by Σ the set of solutions to (P). For convenience later in discussing necessary conditions, let us also define the *singular Lagrangian* L_0:

$$L_0(x, \lambda) := \langle \lambda, h(x) \rangle + r|\lambda| d_C(x).$$

The problem (P) is termed *normal* provided that for any $x \in \Sigma$, the inclusion $0 \in \partial_x L_0(x, \lambda)$ forces $\lambda = 0$. This is the counterpart for our problem of the standard "constraint qualification," or nondegeneracy hypothesis, in mathematical programming which, when f and h are smooth and $C = \mathbb{R}^n$, reduces to the condition that the Jacobian $Jh(x)$ have maximal rank whenever x solves (P). In the following theorem, $M(\Sigma)$ signifies $\bigcup\{M(x) : x \in \Sigma\}$.

THEOREM 1.1. *If (P) is normal, then $\partial V(0)$ is nonempty and we have*

$$\partial V(0) = \operatorname{co}\{\partial V(0) \cap M(\Sigma)\}.$$

This formula establishes a close link between $\partial V(0)$ and the multipliers corresponding to solutions of the problem. Of course, the formula appearing in the theorem is not an exact characterization of $\partial V(0)$ in such terms, since $\partial V(0)$ appears on both sides and we cannot "solve" for it. But the formula gives both an inner and outer estimate. On the one hand, $\partial V(0) \subset \operatorname{co} M(\Sigma)$; on the other hand, $M(\Sigma)$ contains the extreme points of $\partial V(0)$. An exact formula for ∂V in terms of multipliers is not to be expected since, in general, there is a real distinction between optimality (to which V corresponds) and stationarity (necessary conditions in multiplier form). A somewhat sharper formula for $\hat{\partial} V(0)$ actually follows from the proof of the theorem, which is very simple: using little more than the definition of $\partial V(0)$ in terms of proximal normals. And yet the theorem yields a lot of useful consequences. We now proceed to discuss some of them.

COROLLARY 1.1. *$V(\cdot)$ is finite in a neighborhood of 0.*

It suffices to show that $M(\Sigma)$ is bounded, for then, by Theorem 1.1, $\partial V(0)$ is nonempty and bounded, which is equivalent to V being Lipschitz near 0 (see Proposition 1.1). Suppose that, to the contrary, λ_i were an unbounded sequence in $M(\Sigma)$; let $\lambda_i \in M(x_i)$. Then we have

$$0 \in \partial\{f + \langle \lambda_i, h\rangle + r|(1, \lambda_i)|d_C\}(x_i),$$

which implies

$$0 \in \partial\left\{\frac{f}{|\lambda_i|} + \langle \frac{\lambda_i}{|\lambda_i|}, h\rangle + r\frac{|(1,\lambda_i)|}{|\lambda_i|}d_C\right\}(x_i).$$

The compactness hypothesis allows us to extract a subsequence of x_i converging to a point x (which can be shown to belong to Σ by continuity) and also a subsequence of $\lambda_i/|\lambda_i|$ converging to a unit vector λ. In the limit we obtain (using in essence the sum formula (1.10) and the upper semicontinuity of the subdifferential)

$$0 \in \partial L_0(x, \lambda),$$

which contradicts the normality hypothesis, thereby completing the proof.

Corollary 1.1 is a *solvability* conclusion, since it says that for α near 0, there exists at least one solution to the system

$$h(x) + \alpha = 0, \quad x \in C.$$

When $C = \mathbb{R}^n$ and h is smooth, this assertion reduces to the classical result that if h' has maximal rank, then h is locally onto. The present result goes beyond this in allowing h to be merely Lipschitz, and in allowing an "abstract constraint" $x \in C$.

COROLLARY 1.2. *$V(\cdot)$ is Lipschitz near 0.*

As pointed out above, it follows from Theorem 1.1 that $\partial V(0)$ is nonempty and bounded, which is known to imply that V is Lipschitz near 0 (see Proposition 1.1). The formula for $\partial V(0)$ also implies a bound on the local Lipschitz rank:

$$\limsup_{\substack{x\to 0\\ y\to 0}} \frac{|V(x) - V(y)|}{|x - y|} \le \max\{|\lambda| : \lambda \in M(\Sigma)\}.$$

An evident consequence of the theorem is the following condition ensuring that $\nabla V(0)$ exists.

COROLLARY 1.3. *If Σ is a singleton $\{x\}$ and if $M(x)$ is a singleton $\{\lambda\}$, then V is (strictly) differentiable at 0, with $\nabla V(0) = \lambda$.*

The proof consists of noting that as a consequence of the theorem, $\partial V(0)$ reduces to $\{\lambda\}$ in this case.

We can also deduce from the theorem results[12] giving the existence of directional derivatives in the usual sense, as illustrated in the following corollary.

COROLLARY 1.4. *If for each x in Σ, $M(x)$ is a singleton $\{\lambda(x)\}$, then $V'(0;\cdot)$ exists and we have*

$$V'(0;u) = \inf_{x\in\Sigma} \langle u, \lambda(x)\rangle.$$

The proof of this consequence is an exercise in nonsmooth calculus [C1983, p. 243].

Proof of Theorem 1.1. We shall sketch the proof, emphasizing the central idea, which carries over to many other contexts in which a value function appears. The first step is an easy exercise that we omit: to verify that V is lower semicontinuous.

Our goal is to study $\partial V(0)$. Since ∂V is defined in terms of $N_{\operatorname{epi} V}$, the normal cone to $\operatorname{epi} V$, we shall study this object. But it is itself generated by proximal normals to $\operatorname{epi} V$, so we shall study first a typical proximal normal. The point will be that the proximal normal inequality (1.2), when applied to the epigraph of a value function, has the wonderful property of leading to a problem similar to the basic one, but in which the constraint that is being perturbed disappears and gives rise instead to an additional term in the objective function. This is a considerable advantage and leads directly to the multipliers that eventually appear in the theorem.

The analysis is somewhat more direct when couched in terms of proximal subgradients. Accordingly, let β be a proximal subgradient of V at some α, and let x_α be a solution of (P_α). Then by definition, for some $\sigma > 0$ and for all α' near α, we have

$$\begin{aligned} V(\alpha') - \langle \beta, \alpha'\rangle + \sigma|\alpha' - \alpha|^2 &\geq V(\alpha) - \langle \beta, \alpha\rangle \\ &= f(x_\alpha) + \langle \beta, h(x_\alpha)\rangle. \end{aligned}$$

Let us substitute $-h(x)$ for α' in this inequality, where x is any point in C, sufficiently near x_α (whence α' is near α). Observing that $V(\alpha')$ is necessarily majorized by $f(x)$, we deduce that for all x in $\widetilde{C} := C \cap \Omega$, where Ω is a closed neighborhood of x_α, we have

$$f(x) + \langle \beta, h(x)\rangle + \sigma|h(x) - h(x_\alpha)|^2 \geq f(x_\alpha) + \langle \beta, h(x_\alpha)\rangle,$$

that is, the function $\tilde{f}$ defined as the left side of this inequality has a minimum over $\widetilde{C}$ at x_α. Now the Lipschitz rank in a neighborhood of $\widetilde{C}$ of the function $(1,\beta)\cdot[f,h]$ does not exceed $|(1,\beta)|K$ by assumption, while $|h(x) - h(x_\alpha)|^2$ has arbitrarily small Lipschitz rank if x is sufficiently near x_α. Thus by restricting Ω

[12] Gauvin and Janin [GJ1988] describe several other issues and their analyses.

further if necessary we can arrange for $\tilde{f}$ to have Lipschitz rank at most $|(1,\beta)|r$. It follows from Proposition 1.3 (§1.3) that x_α is a local (unconstrained) minimizer of the function

$$f(x) + \langle \beta, h(x)\rangle + \sigma|h(x) - h(x_\alpha)|^2 + r|(1,\beta)|d_{\widetilde{C}}(x).$$

Since the function $|h(x) - h(x_\alpha)|^2$ has subdifferential $\{0\}$ at x_α (exercise), and in view of the fact that $\partial d_{\widetilde{C}}(x_\alpha) = \partial d_C(x_\alpha)$ (since d_C and $d_{\widetilde{C}}$ agree near x_α), we obtain

$$0 \in \partial_x L(x_\alpha, \beta).$$

Thus β is a multiplier corresponding to x_α.

Let us now consider sequences β_i, where β_i is a proximal subgradient of V at α_i and $\alpha_i \to 0$. We have just shown that β_i belongs to $M(x_i)$, where x_i solves (P_{α_i}). It follows as in the proof of Corollary 1.1 that there is a uniform bound on $M(x)$ whenever x solves (P_α) and α is sufficiently close to 0. Thus V is Lipschitz near 0 by Proposition 1.2, and $\hat{\partial}V(0)$ is a nonempty compact set consisting of all points λ of the form $\lambda = \lim \beta_i$, where the sequence β_i is as given above. By compactness, we may assume that the corresponding x_i converge to a limit x; it follows easily that $x \in \Sigma$. By passing to the limit in the multiplier relation, we obtain $\lambda \in M(x) \subset M(\Sigma)$. Hence $\hat{\partial}V(0)$ is a subset of $M(\Sigma)$. The proof is complete, since this formula is easily seen to imply the one in the statement of the theorem. □

Remark. The main point made in the proof is the ease with which ∂V can be estimated if we approach the problem via proximal analysis; the moral carries over to many other optimization problems, some of which will be discussed in Chapter 4 . In that and other chapters, the exposition is made essentially in terms of the subdifferential, so we have couched Theorem 1.1 in those terms. It is worth pointing out, however, that the argument needs no modification whatever when $\hat{\partial}$ replaces ∂. Thus if we define the multiplier set $\hat{M}(x)$ as those λ satisfying $0 \in \hat{\partial}_x L(x,\lambda)$, the proof as given yields the formula

$$\hat{\partial}V(0) \subseteq \hat{M}(\Sigma),$$

which is a tighter estimate than the previous statement, since we evidently have $\hat{M}(x) \subseteq M(x)$. And the normality hypothesis is reduced: it suffices to suppose that no nonzero λ satisfies $0 \in \hat{\partial}_x L_0(x,\lambda)$ when $x \in \Sigma$. For a further remark in this vein, see Theorem 4.1 (footnote).

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CHAPTER 2

The Basic Problem in the Calculus of Variations

2.1. Some background.

Our first look at dynamic optimization is in the calculus of variations, a field that is approximately three centuries old, yet continues to be a source of challenging mathematical issues as well as a context for new and interesting applications. The problem that we shall focus on is the simplest; it involves a *Lagrangian* L, namely, a function mapping $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$, where T is a fixed positive scalar defining the horizon of the problem. The three variables (t,s,v) upon which L depends are usually identified as time, state, and velocity. The functional Λ associates with a certain class X of mappings $x : [0,T] \longrightarrow \mathbb{R}^n$ the "integral of L along x":

$$\Lambda(x) := \int_0^T L(t, x(t), \dot{x}(t))\, dt.$$

Here $\dot{x}$ designates dx/dt. The problem (P) that we study is the following:

$$(P) \qquad \begin{array}{l} \text{minimize } \Lambda(x) \text{ over } X \\ \text{subject to } x(0) = x_0,\ x(T) = x_T. \end{array}$$

The points x_0, x_T entering into the boundary condition are given. The choice of the class X will play an important role in the discussion. In early work on the subject, X is a class of smooth functions, like C^2; let us adopt this choice for now.

The first systematic effort to study (P) is due to Euler, who published in 1744 the basic necessary condition for optimality. Let us sketch the standard argument now used to derive it. Let x solve (P), and consider any C^2 function $h : [0,T] \longrightarrow \mathbb{R}^n$ vanishing at 0 and at T; such a function is called a *variation*. Let λ be a scalar, and observe that $x + \lambda h$ is an element of C^2 satisfying the boundary constraints of (P). Consequently, since x solves (P), we have $\Lambda(x + \lambda h) \geq \Lambda(x)$.

In other terms, the function $g(\lambda)$ defined by $g(\lambda) := \Lambda(x+\lambda h)$ admits a minimum at $\lambda = 0$. So if $g'(0)$ exists, it must be zero.

Under well-known conditions, $g'(0)$ exists and can be calculated by "differentiating under the integral" with respect to λ the following expression:

$$g(\lambda) := \int_0^T L(t, x + \lambda h, \dot{x} + \lambda \dot{h})\, dt$$

and setting $\lambda = 0$. When $X = C^2$, the argument can be justified when L is twice continuously differentiable, which we suppose to be the case until further notice. The calculation gives

$$\int_0^T \{L_s(t, x, \dot{x})h + L_v(t, x, \dot{x})\dot{h}\}\, dt = 0. \tag{2.1}$$

Let us now integrate the second term by parts, bearing in mind that $h(0) = h(T) = 0$. We derive

$$\int_0^T \left\{L_s(t, x, \dot{x}) - \frac{d}{dt}L_v(t, x, \dot{x})\right\} h(t)\, dt = 0.$$

Note that this obtains for any variation h, and that the quantity in braces is independent of h. It is tempting, but nevertheless correct, to conclude that this quantity vanishes identically:

$$\frac{d}{dt}L_v(t, x(t), \dot{x}(t)) = L_s(t, x(t), \dot{x}(t)). \tag{2.2}$$

This is the *Euler equation.* A function x satisfying it is called an *extremal.* We shall refer to the argument above as *the standard variational argument.*

The Euler equation is essentially a second-order differential equation, and the boundary conditions account for two degrees of freedom, so in many cases a unique admissible extremal exists, although this is not true generally. In elementary books, (P) is "solved" in sample problems by determining the extremals. For example, consider the problem

$$\text{minimize} \int_0^1 |\dot{x}(t)|^2\, dt : x \in C^2,\ x(0) = 0,\ x(1) = 1. \tag{2.3}$$

This is the special case of (P) in which $n = 1$, $T = 1$, $x_0 = 0$, $x_1 = 1$, and $L(t, s, v) = v^2$. The Euler equation (2.2) becomes

$$\ddot{x}(t) = 0, \tag{2.4}$$

which has a unique solution in C^2 satisfying the boundary conditions, the function $\hat{x}(t) \equiv t$. So $\hat{x}$ is the solution (provided one exists when $X = C^2$, but we shall return to this point).

Another fundamental necessary condition for optimality is due to Weierstrass. Under our standing smoothness hypotheses, this asserts that if x solves (P), then we have, for each t in $[0, T]$,

$$L(t, x(t), \dot{x}(t) + v) - L(t, x(t), \dot{x}(t)) \geq \langle v, L_v(t, x(t), \dot{x}(t))\rangle \quad \text{for all } v \in \mathbf{R}^n. \tag{2.5}$$

Of the several ways of interpreting this, the one most relevant to us is in terms of convexity. The *convex hull* of the function $L(t, x(t), \cdot)$ is by definition the largest convex function that it majorizes; we designate it by $\widetilde{L}(t, x(t), \cdot)$. Now it can be shown that condition (2.5) is equivalent to the following statement: L and $\widetilde{L}$ agree along x (i.e., at points of the form $(t, x(t), \dot{x}(t))$). Let us restate this another way, in terms of the version $(\widetilde{P})$ of (P) obtained by replacing L by $\widetilde{L}$:

$$(\widetilde{P}) \qquad \begin{aligned} &\text{minimize } \int_0^T \widetilde{L}(t, x(t), \dot{x}(t))\, dt \text{ over } X \\ &\text{subject to } x(0) = x_0, x(T) = x_T. \end{aligned}$$

We refer to $\widetilde{L}$ as the *relaxed Lagrangian* and to $(\widetilde{P})$ as the *relaxed problem*. Then (2.5) (for each $t \in [0, T]$) is equivalent to the assertion that the solution x to the original (P) is also a solution to the relaxed problem $(\widetilde{P})$.[13]

For completeness, let us mention the only remaining major necessary condition for (P), which is due to Jacobi. We need the concept of *conjugacy*: the point τ in $(0, T]$ is said to be conjugate to 0, relative to a given extremal $x(\cdot)$, if there exists a nontrivial solution of the following two-point boundary value problem:

$$(2.6) \qquad \frac{d}{dt}\{L_{vv}\dot{h} + L_{vx}h\} - L_{xv}\dot{h} - L_{xx}h = 0, \qquad h(0) = h(\tau) = 0$$

where the derivatives of L are evaluated along x. Jacobi's necessary condition asserts that any solution x to (P) admits no points in $(0, T)$ conjugate to zero. This condition is of second order; it corresponds to the fact that the function g that was defined in connection with the standard variational argument satisfies $g''(0) \geq 0$.

One of the principal goals (and achievements) of the calculus of variations in the nineteenth century was the development of sets of sufficient conditions having the form of suitably strengthened necessary conditions. These are inevitably of local type. An example of such a result, essentially due to Weierstrass, is the following. Let x in C^2 be an extremal admitting no points in $(0, T]$ conjugate to 0, and suppose that the Lagrangian $L(t, s, v)$ is convex as a function of v (for each (s, t)) and that we have $L_{vv} > 0$ along x. Then x is a *strong local minimum* for (P); i.e., for some $\epsilon > 0$, for all $y \in C^2$ satisfying, together with the boundary constraints, the condition $|y(t) - x(t)| < \epsilon$ (for $t \in [0, T]$), we have $\Lambda(y) \geq \Lambda(x)$.

2.2. Enter nonsmoothness.

It is not necessarily the case that smoothness of the Lagrangian is inherited by the solution to (P). Take for example $n = 1$ and $L(t, s, v) = s^2(v - 1)^2$; consider the version of (P) on $[0, 2]$ that imposes $x(0) = 0$, $x(2) = 1$. Now let $\hat{x}$ be the function that is equal to 0 on $[0, 1]$ and to $t - 1$ for $t \in [1, 2]$. Then $\Lambda(\hat{x}) = 0$; since L is nonnegative, $\hat{x}$ solves (P) whenever the class X contains $\hat{x}$. Furthermore, it is easy to see that $\Lambda(x) > 0$ for any *smooth* admissible x, yet the infimum of Λ over such x is 0.

[13]This is a case of "admissible relaxation," the details of which appear in [C1975c].

Considerations such as these have led to choices of X that encompass some nonsmooth functions, like $\hat{x}$ above. A class that is frequently used is *PWS*, the class of piecewise-smooth functions. A continuous x on $[0,T]$ belongs to *PWS* if there is a partition $0 = t_0 < t_1 \cdots < t_k = T$ of $[0,T]$ such that x is C^1 on each (t_i, t_{i+1}), with finite one-sided directional derivatives at each partition point.

With the choice $X = PWS$, some results are obtainable in essentially the same way as before. In particular, the standard variational argument that led to the Euler equation (2.2) continues to be justifiable. We obtain that (2.2) holds except at corner points of x. Note, however, that information is lost. For example, (2.3) has Euler equation (2.4) (at noncorners) as before, but in *PWS* there are infinitely many solutions of (2.4) satisfying the boundary conditions, in contrast to the unique solution in the class C^2.

The *first Erdmann condition* (1877) was developed to supplement the Euler equation when nonsmooth x are present. It affirms that if $x \in PWS$ solves (P), then the function $t \to L_v(t, x(t), \dot{x}(t))$ is continuous (i.e., admits removable discontinuities at corner points of x). In (2.3), this amounts to the statement that $\dot{x}$ is continuous, in other words, that any solution in *PWS* must actually be smooth. This is our first instance of a *regularity theorem*, and already the intimate relationship with necessary conditions (which we will study later), is evident.

In 1879 duBois–Reymond found a way to incorporate the first Erdmann condition into a stronger form of the Euler equation. To see how to derive it, consider again the standard variational argument, up to the point where (2.1) has been obtained. Instead of proceeding to integrate the second term by parts, choose the first. The result is

$$\int_0^T \left\{ L_v(t,x,\dot{x}) - \int_0^t L_s(\tau,x,\dot{x})d\tau \right\} \dot{h}(t)\,dt = 0.$$

Since h is any variation, we may again be tempted to conclude that the quantity in braces vanishes. This is not quite right: as the duBois–Reymond lemma states, the correct conclusion is that the quantity in braces is constant. To summarize, we have

$$L_v(t,x(t),\dot{x}(t)) = c + \int_0^t L_s(\tau,x(\tau),\dot{x}(\tau))d\tau, \tag{2.7}$$

which is the *integral form* of Euler's equation. Note that (2.7) implies the original (differentiated) form (2.2), as well as the first Erdmann condition. In general, however, we cannot pass from (2.2) to (2.7) unless we know, for example, that $\dot{x}$ is continuous.

Another instance in which nonsmoothness creates a distinct necessary condition is provided by the *second Erdmann condition*, which, when L is *autonomous* (i.e., independent of t, the time variable), asserts

$$\langle L_v(x(t),\dot{x}(t)),\dot{x}(t)\rangle - L(x(t),\dot{x}(t)) = \text{constant}.$$

When L and x are sufficiently smooth (C^2), the left side above has zero derivative as a consequence of the Euler equation (exercise), so the condition can be

considered an immediate corollary of the Euler equation. When x is not smooth however, it is a distinct conclusion and must be derived independently.

The Weierstrass condition (2.5) also adapts to $X = PWS$; however, the Jacobi condition and consequently the sufficiency results based on it are difficult to extend to such a setting.[14]

Another type of nonsmoothness that comes to mind, but which played little or no role in classical developments, involves the possibility of the Lagrangian itself being less smooth than we have been supposing. This sort of consideration is much more modern in nature. However, even in our discussion above it did enter implicitly when we defined the relaxed Lagrangian $\widetilde{L}$. It is possible for $\widetilde{L}$ to fail to be C^2 even when L is C^∞. (We leave as an exercise the proof that this is the case for $L = (v^2 - 1)^2$.) A specific application involving a nonsmooth Lagrangian appears in §2.8, where we discuss the dual action principle for periodic Hamiltonian trajectories, with the help of an extension of the Euler equation to such cases presented in §2.5.

There are reasons other than those alluded to above as to why extensions to classes of nonsmooth functions such as *PWS* remained unsatisfactory in some respects. The principal reason is the lack of an existence theory.

2.3. Enter Tonelli.

The *deductive method* in optimization (*deductive*: from the general to the particular) proceeds by the following logical chain of reasoning:

(A) A solution to the problem exists.
(B) The necessary conditions are applicable, and they identify certain candidates (extremals).
(C) Further elimination (if necessary) identifies the solution x.

This appealing argument, familiar from elementary calculus (set $f'(x) = 0$, find the critical points, etc.), depends on the existence of a solution being assured (step A). While this point is not always stressed in elementary calculus, or can be dealt with on an ad hoc basis, it is considerably subtler in the calculus of variations, where what might appear to be very reasonable problems can fail to admit solutions. And yet no existence theorems could be proven in the calculus of variations until Tonelli (1915) (with one exception: Hilbert's somewhat earlier special parametric existence theorem). This is why in the classical theory such stress is laid for some two centuries on necessary conditions and then on their strengthening to become (locally) sufficient. Although necessary conditions usually played an important role in identifying tentative solutions, one was then faced with the problem of proving that one indeed had a solution. Thus the classical approach, in the absence of existence theorems, is the *inductive method* (*inductive*: from the particular to the general): identify a candidate x through fair means or foul (intuition, possibly questionable necessary conditions, trial and error, $\cdots$); then prove that x is as good (at least locally) as any other candidate. An example of this method in elementary calculus would be to show that

[14] A discussion of this point appears in [CZ1986].

for a given x, a function $f(y)$ can be expressed in the form $c + [g(y) - g(x)]^2$; it follows then that x minimizes f. This simple example has more bearing in the calculus of variations than we might think, since the idea behind it is very similar to Caratheodory's method. This inductive method (explained in Chapter 3), suitably extended via nonsmooth analysis, can be considered as the basis of all the principal sufficient conditions and fills the gap alluded to above with respect to sufficient conditions in the absence of classical smoothness requirements.

We turn now to Tonelli's existence theory. He termed the approach of immediately proving the existence of a solution the *direct method.* It begins with a set of hypotheses on the Lagrangian $L(t, s, v)$:

(T1) L is twice continuously differentiable.

(T2) L is (quadratically) coercive. There exist constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$L(t, s, v) \geq \alpha |v|^2 + \beta \quad \text{for all } (t, s, v).$$

(T3) $L_{vv}(t, s, v) \geq 0 \quad$ for all (t, s, v).

In (T3), L_{vv} refers to the $n \times n$ Hessian matrix of second partial derivatives, and ≥ 0 signifies positive semidefinite. When L is C^2 in v, (T3) is equivalent to L's being convex in v. This is really not an unreasonable hypothesis, partly because experience shows that the important classes of applications already feature this property, and also because (under mild assumptions) the problem (P) and its relaxed form $(\widetilde{P})$ defined earlier are equivalent, and the relaxed Lagrangian $\widetilde{L}$ is always convex in v by construction. (It may not be C^2, however, which is one reason for wishing to weaken the smoothness requirements on L, as will be done.)

We shall discuss presently the role of these hypotheses in the proof of the theorem; but equally seminal was the choice of class X which Tonelli made: he took $X = AC$, the class of absolutely continuous functions.[15] We term *arc* an absolutely continuous map from $[0, T]$ to $\mathbb{R}^n$. Tonelli's introduction of arcs was a radical departure from earlier work, since AC is a much bigger class than PWS, for example. In particular, an arc need not have an essentially bounded derivative.

From now on when we refer to problem (P), it is understood that AC is the choice of X.

TONELLI'S EXISTENCE THEOREM. *When* (T1)–(T3) *hold, when* $X = AC$, *then a solution to* (P) *exists.*

The proof is easily summarized in modern terms, which relate to concepts that were just being appreciated in Tonelli's time. A minimizing sequence $\{x_i\}$ is chosen; the coercivity (T2) assures that the sequence of derivatives $\{\dot{x}_i\}$ is bounded in L^2; weak compactness assures the existence of a subsequence of $\{\dot{x}_i\}$ (we do not relabel) converging weakly to an element of L^2; this weak limit is identified as the derivative of an arc x to which the x_i converge uniformly;

[15] A function $x : [0, T] \longrightarrow \mathbb{R}^n$ is absolutely continuous provided that for some Lebesgue integrable function v, x is expressible as $x(t) = c + \int_0^t v(\tau) d\tau$. It follows then that $\dot{x} = v$ almost everywhere.

the regularity of L provided by (T1) together with the convexity of L in the velocity provided by (T3) assure that the functional Λ is lower semicontinuous with respect to this convergence. It follows that x solves (P).

Although the hypotheses of the theorem can be relaxed somewhat, the theorem itself remains even today the central existence theorem for problems such as (P). Tonelli's direct method of proof has been applied and adapted to countless other settings. A weaker set of hypotheses under which the proof goes through without major modification would be, for example (with $X = AC$ as before),

(T1)′ L is locally Lipschitz.

(T2)′ L is *coercive*. For some $\theta : [0, \infty) \to \mathbb{R}$ with $\lim_{r\to\infty} \theta(r)/r = +\infty$, we have

$$L(t, s, v) \geq \theta(|v|) \quad \text{for all } (t, s, v).$$

(T3)′ $L(t, s, \cdot)$ is convex for each (t, s).

As we have said, Tonelli's great leap forward hinged upon working within a much larger class of functions (AC) than had ever been used before. The use of this class did seem to make a difference. For example, the standard variational argument used to derive the Euler equation can no longer be rigorously justified when x lies in AC (rather than C^2 or PWS, for example). The argument *can* be defended when x has bounded derivative, i.e., when x is Lipschitz. Is this just a technical point, or could there be a real difference in the resulting problems? Specifically, we are led to ask the following natural questions:

(Q1) To what extent is the choice $X = AC$ necessary in Tonelli's Existence Theorem? Does the solution x actually have bounded derivative in every case?

(Q2) Could the infimum in (P) when $X = AC$ be strictly less than the infimum when $X = PWS$ or C^2?

(Q3) What necessary conditions hold under just the hypotheses of Tonelli's Existence Theorem? In particular, do we still have the Euler equation in integral form?

The second question was the first to be answered. In 1926 Lavrentiev adduced an example of (P) (with $n = 1$) for which we have (for the same given boundary conditions)

$$\inf_{x\in AC} \Lambda(x) < \inf_{x\in C^2} \Lambda(x). \tag{2.8}$$

It is easy to show that when L is continuous, this latter infimum coincides with the one over PWS or over the class of Lipschitz functions. Thus when the Lavrentiev phenomenon (2.8) occurs, the solution to (P) (with $X = AC$) certainly has an unbounded derivative. A somewhat simpler example in which the Lavrentiev phenomenon occurs was given later by Maniá (see [CES1983, p.514]), with $n = 1$, $T = 1$, Lagrangian $L = (s^3 - t)^2 v^6$, and boundary conditions $x(0) = 0$, $x(1) = 1$. It is apparent that the solution to the problem is $x(t) = t^{1/3}$, an arc that fails to have a bounded derivative. This example is *not* an answer to question (Q1), however, since the Lagrangian is not coercive. We seek a case of (P) in the context of Tonelli's Existence Theorem in which the solution is not Lipschitz.

A candidate for such an example was first proposed by Ball and Mizel. It has $T = 1$, $n = 1$, and Lagrangian

$$L(t, s, v) = rv^2 + (s^3 - t^2)^2 v^{14},$$

where r is a positive constant. As they pointed out, the arc $x(t) = kt^{2/3}$ satisfies the pointwise Euler equation (2.2) almost everywhere, when r and k are related algebraically in a certain way. We shall mention presently a result of Tonelli's implying that any solution to this problem must be at least an extremal in this sense. Here is a situation for the inductive method. The Euler equation is difficult to analyze directly, in particular to the point of excluding other solutions, yet a suspect extremal exists whose optimality is to be confirmed. The first proof that the x above is in fact the unique global solution was given in [CV1984]. The proof is based upon the method of verification functions and will be explained in Chapter 3.

The Lavrentiev phenomenon (2.8) has significant implications for the numerical calculation of $\inf(P)$, for the usual methods proceed by approximation involving a class of Lipschitz functions (power series expansions, piecewise linear approximants, splines). When (2.8) holds, such approximations will be strictly bounded away from the infimum over AC, even if they appear to converge. Thus a goal to pursue is to identify conditions that serve to exclude the Lavrentiev phenomenon. Again, regularity plays a role here: if we know that the solution x to (P) (with $X = AC$) has a bounded derivative, then (2.8) cannot occur.[16]

The integral form of the Euler equation is not satisfied by the solution to the Ball–Mizel problem, which answers the second part of (Q3). Tonelli himself gave a partial answer to the first part of (Q3) via the following result, which again illustrates the close connection between regularity and necessary conditions.

TONELLI'S REGULARITY THEOREM. *Let $n = 1$ and let L be twice continuously differentiable, with $L_{vv} > 0$. Then any solution x to (P) admits an open subset Ω of full measure in $[0, T]$ in which x is locally Lipschitz.*

This has a corollary that bears upon (Q3). Let τ in $(0, T)$ be any point in the set Ω of the theorem, and let $\epsilon > 0$ be such that x is Lipschitz on $(\tau - \epsilon, \tau + \epsilon)$. Now x, as a solution to (P), must also solve the subproblem

$$\text{minimize} \int_{\tau-\epsilon}^{\tau+\epsilon} L(t, y, \dot{y})\, dt \text{ over arcs } y \text{ on } [\tau - \epsilon, \tau + \epsilon]$$
$$\text{satisfying } y(\tau - \epsilon) = x(\tau - \epsilon),\ \ y(\tau + \epsilon) = x(\tau + \epsilon).$$

(We leave this exercise in reasoning, sometimes referred to as "the principle of optimality," to the reader.) But on the restricted interval x is Lipschitz, and so the standard variational argument applied to the above subproblem shows that x satisfies the differentiated Euler equation (2.2) on that interval. It follows that x satisfies (2.2) almost everywhere in $[0, T]$. This sheds some light on (Q3) for $n = 1$.

[16] A recent paper of Ball and Knowles [BK1987] describes their efforts to solve numerically problems exhibiting the Lavrentiev phenomenon. We also refer the reader to the discussions in [BM1985] and [L1987b].

Necessary conditions for (P) with $X = AC$ and for $n > 1$ do exist, but these postulate growth conditions on the Lagrangian beyond those of Tonelli's Existence Theorem. For example, the classical Tonelli–Morrey condition

$$|L_s| + |L_v| \leq k|L| + r \qquad (k > 0) \tag{2.9}$$

has been used. Under it, the solution to (P) can be shown to have a bounded derivative and to satisfy the Euler equation in integral form. Note, however, that (2.9) can fail even for polynomial Lagrangians, as is the case (necessarily) for the Ball–Mizel Lagrangian cited earlier.

The search then for verifiable conditions under which we have the regularity of the solution (to the extent, say, of its being Lipschitz), or under which the Lavrentiev phenomenon is excluded, or under which strong necessary conditions hold, reveals itself as an important line of study, a direct legacy of Tonelli's work.

2.4. A regularity theorem.

We shall present in this section a recent regularity theory for (P) due to Clarke and Vinter.[17] We need the concept of a *regular point* τ of an arc x. This is a point τ in $[0,T]$ such that

$$\liminf_{\substack{0\leq s\leq\tau\leq t\leq T \\ s,t\to\tau \\ s\neq t}} \frac{|x(t)-x(s)|}{|t-s|} < \infty. \tag{2.10}$$

Equivalently, τ is regular if there exists some bounded sequence of difference quotients whose base points in $[0,T]$ bracket τ and converge to it. Since an arc is differentiable almost everywhere, it follows that almost all points τ are regular.

If the lim inf in (2.10) is replaced by lim sup, then the resulting condition says that x is Lipschitz in a neighborhood of τ. For a general arc x, this is a much stronger condition than (2.10). But for arcs that are solutions of (P), the two conditions turn out to be equivalent. That is the content of the following theorem, whose hypotheses are (T1)′–(T3)′.

THEOREM 2.1. *Let x be a solution to (P), and let τ be a regular point of X. Then there is an interval open in $[0,T]$ containing τ in which x is locally Lipschitz.*

An immediate consequence of the theorem, proven just as in the case of Tonelli's Regularity Theorem for $n = 1$, is that x is a "pointwise extremal." This is now obtained for $n > 1$ and under reduced smoothness hypotheses on L.

COROLLARY 2.1. *Let L be C^1, in addition to the hypotheses of Theorem 2.1. Then almost everywhere on $[0,T]$ the Euler equation (2.2) holds.*

The theorem does not say, of course, that x is locally Lipschitz on $[0,T]$. This is fortunate, since we know that conclusion to be false in general. (It does follow that there is an open set Ω of full measure in $[0,T]$ in which x has that property.) As will be seen, however, the theorem can serve as a wedge toward that stronger conclusion (i.e., $\Omega = [0,T]$). The theorem does provide a foothold inside $[0,T]$ in which x is Lipschitz and which, if certain additional properties are present,

[17] The results of this section are proven under weaker hypotheses in [CV1985].

can then be extended left and right until x is seen to be Lipschitz on all of $[0,T]$. The proof of the following corollary[18] will illustrate the technique. We omit the proof of Theorem 2.1 itself; it is sufficiently complex that even imparting the general idea is inappropriate here. Nonsmooth analysis again plays a major role.

COROLLARY 2.2. *If (P) is autonomous, then every solution x is Lipschitz.*

Proof. We begin by choosing a regular point τ in $(0,T)$. By the theorem, there is a subinterval $[\tau,\beta]$ of $[\tau,T]$ of positive measure in which x is Lipschitz. We let β_0 be the supremum of all such β. We now proceed under the assumption that L is smooth, for simplicity, and to avoid for now the need to invoke necessary conditions for nonsmooth Lagrangians. We claim that β_0 must be T. Let β_i be an increasing sequence in (τ,β_0) converging to β_0. Note that x solves the subproblem

$$\begin{aligned} &\text{minimize } \int_\tau^{\beta_i} L(y,\dot{y})\,dt \\ &\text{subject to } y(\tau)=x(\tau),\ y(\beta_i)=x(\beta_i), \end{aligned}$$

and further that x is Lipschitz in $[\tau,\beta_i]$. Therefore, the usual necessary conditions are satisfied, in particular, Erdmann's second condition:

$$\langle \dot{x}, L_v(x,\dot{x})\rangle - L(x,\dot{x}) = c_i = \text{constant} \quad \text{a.e.}, \qquad t\in[\tau,\beta_i].$$

It follows, of course, that the constant is the same $(=c)$ for every i, and that the equation above holds on $[\tau,\beta_0)$. But this condition, together with the coercivity (T2)′, implies an essential bound for $|\dot{x}(t)|$, $t\in[\tau,\beta_0]$ (exercise). It follows that β_0 is a regular point. But then, by the theorem, a neighborhood of β_0 in $[0,T]$ exists in which x is Lipschitz. By definition of β_0, it must be the case that $\beta_0 = T$. □

The proof technique of Corollary 2.2 can be adapted to prove the following further consequences of the theorem, the first of which generalizes Corollary 2.2 itself. Again we present only the case in which L is smooth, for simplicity. The nonsmooth case, and further consequences, are given in [CV1985].

COROLLARY 2.3. *Suppose that L satisfies the growth condition*

$$|L_t(t,s,v)| \le c|L(t,s,v)| + k(t),$$

where k is integrable. Then any solution to (P) is Lipschitz.

COROLLARY 2.4. *Suppose that L satisfies the growth condition*

$$|L_s(t,s,v)| \le c|L(t,s,v)| + k(t)|L_v(t,s,v)| + m(t), \tag{2.11}$$

where k and m are integrable. Then any solution to (P) is Lipschitz.

The growth condition in Corollary 2.4 may be considered a generalization of the classical Tonelli–Morrey condition (2.9) (note the transposition of the $|L_v|$ term to the right side, making the condition less difficult to satisfy), while that of Corollary 2.3 is of a new type. In the case when L is not C^1, it is expressed in terms of subdifferentials, as follows:

$$|\zeta| \le c|L(t,s,v)| + k(t) \quad \forall \zeta \in \partial_t L(t,s,v).$$

[18]This result first appeared in [CV1985]; see also [AAB1989] for extensions.

Recently, Richard Vinter has pointed out to us another situation in which the theorem can be used to deduce Lipschitz regularity.

COROLLARY 2.5. *Suppose that for each t, the function $L(t,\cdot,\cdot)$ is convex. Then any solution x to (P) is Lipschitz.*

Proof. Let τ, β_0, β_i be chosen as in the proof of Corollary 2.2, and assume again that L is smooth, to simplify the discussion. Let $p(t)$ signify $L_v(t,x(t),\dot{x}(t))$. Then the Euler equation applies on $[\tau,\beta_i]$. It is equivalent to $(\dot{p},p) = \nabla_{s,v}L(t,x,\dot{x})$ almost everywhere, whence (by convexity)

$$L(t,y,v) - L(t,x,\dot{x}) \geq \langle y-x,\dot{p}\rangle + \langle v-\dot{x},p\rangle \quad \forall (y,v).$$

Take $y = x(t)+u$, $v=0$, where u is any unit vector, to deduce

$$|\dot{p}(t)| = \max_{|u|\leq 1}\langle u,\dot{p}(t)\rangle \leq \langle \dot{x}(t),p(t)\rangle + \max_{|u|\leq 1} L(t,x(t)+u,0) - L(t,x(t),\dot{x}(t)).$$

It follows from this that p satisfies a condition of the form

$$|\dot{p}(t)| \leq \alpha(t)\,|p(t)| + \gamma(t)$$

for certain integrable functions α and γ not depending on i. Now Gronwall's lemma leads to a uniform bound for $p(t)$, which equals $L_v(t,x(t),\dot{x}(t))$. Since $|L_v(t,x,v)|$ goes to $+\infty$ with $|v|$ (by coercivity), this implies a uniform bound on $|\dot{x}(t)|$. As in the proof of Corollary 2.2, this leads to the desired conclusion. □

These corollaries constitute the principal known verifiable conditions assuring that solutions to (P) are Lipschitz, and hence that the Lavrentiev phenomenon does not occur, and that the Euler equation in integral form is valid.[19] Once the watershed of x's being Lipschitz is crossed, further regularity conclusions regarding x are readily obtainable from essentially classical arguments.

PROPOSITION 2.1. *Suppose that $L(t,s,v)$ satisfies* (T1)′ *(without necessarily being C^1) and that x is a Lipschitz solution of (P).*

(i) *If $L(t,x(t),\cdot)$ is strictly convex for each t, then x is C^1.*

(ii) *If in addition L is C^r for $r\geq 2$ and $L_{vv}(t,x(t),\dot{x}(t)) > 0$, then x is C^r.*

We shall sketch only the proof of (ii), a classical argument of Hilbert. We begin by writing the Euler equation in integral form, which is known to be a valid necessary condition because x is Lipschitz:

$$L_v(t,x,\dot{x}) = c + \int_0^t L_s(\tau,x,\dot{x})d\tau.$$

The right side is C^1 in t, and so is the function $(t,v) \longrightarrow L_v(t,x(t),v)$. It follows from the Implicit Function Theorem that $\dot{x}$ is C^1, i.e., that x is C^2. This allows us to differentiate through above and solve to get

$$\ddot{x} = [L_{vv}]^{-1}\{L_s - L_{vt} - L_{vs}\dot{x}\},$$

[19]Further results related to Theorem 2.1 include a refinement in the case that L is essentially a polynomial [CV1986], and an extension to the isoperimetric problem [C1987a]. There is a vast literature on variational theory for the case of multiple integrals, which we do not treat; see for example [G1983].

where the derivatives of L are evaluated along x. Now suppose x is known to be C^{r-1} and L is C^r $(r \geq 2)$. The right side of the last equation is readily seen to be C^{r-2}, hence so is $\ddot{x}$, i.e., x is C^r. □

2.5. The Euler inclusion.

The extension of problem (P) to arcs made a satisfying existence theory possible. As we now see, it is also a useful step in obtaining necessary conditions for problems in which the Lagrangian is nonsmooth. Note that (P) makes perfect sense even if L is nondifferentiable, for example, if it is assumed locally Lipschitz.

How can the Euler equation be extended to such cases? The key is to return to the smooth setting and rewrite the Euler equation in terms of the arc $p(t)$ defined by the right side of (2.7). Then p satisfies

$$(\dot{p}(t), p(t)) = \nabla_{s,v} L(t, x(t), \dot{x}(t)) \quad \text{a.e.}$$

This suggests the following appropriate generalization.

THEOREM 2.2. *Let L be locally Lipschitz, and suppose that the Lipschitz arc x solves (P). Then there exists another Lipschitz arc p such that*

$$(\dot{p}(t), p(t)) \in \partial L(t, x(t), \dot{x}(t)) \quad \text{a.e.}, \tag{2.12}$$

where the subdifferential ∂L is taken with respect to (s, v).

Proof. There is some interest in seeing how the standard variational argument can be adapted with the tools of nonsmooth analysis.[20] We begin, just as in §2.1, with a variation h and the observation that for any positive scalar λ we have $\Lambda(x + \lambda h) \geq \Lambda(x)$. This yields

$$\begin{aligned} 0 &\leq \liminf_{\lambda \downarrow 0} \int_0^T \frac{L(t, x + \lambda h, \dot{x} + \lambda \dot{h}) - L(t, x, \dot{x})}{\lambda}\, dt \\ &\leq \int_0^T \limsup_{\lambda \downarrow 0} \frac{L(t, x + \lambda h, \dot{x} + \lambda \dot{h}) - L(t, x, \dot{x})}{\lambda}\, dt \\ &\leq \int_0^T L^0(t, x, \dot{x}; h, \dot{h})\, dt \end{aligned}$$

(see (1.6); the directional derivative is taken with respect to the (s, v) variables)

$$\begin{aligned} &= \int_0^T \max\{(h, \dot{h}) \cdot \partial L(t, x, \dot{x})\}\, dt \quad \text{(by (1.5))} \\ &= \max_{(\zeta,\psi) \in A} \int_0^T \langle (h, \dot{h}), (\zeta, \psi) \rangle\, dt, \end{aligned}$$

where A is the set of measurable selections of the multifunction $t \to \partial L(t, x(t), \dot{x}(t))$. The conclusion to this point is that we have

$$\min_h \max_{(\zeta,\psi) \in A} \int_0^T \langle (h, \dot{h}), (\zeta, \psi) \rangle\, dt = 0.$$

[20] The details of this argument are given in [C1977].

We can invoke a minimax theorem to switch the min and the max and then conclude that for some element (ζ, ψ) of A we have

$$\int_0^T \langle (h, \dot{h}), (\zeta, \psi) \rangle dt = 0 \quad \text{for all variations } h.$$

Now the duBois–Reymond lemma applies to yield the conclusion that ψ is equal to an arc p almost everywhere, with $\dot{p} = \zeta$ almost everywhere. That is, p satisfies (2.12). □

Remark. Problem (P) can be extended to more general boundary terms and constraints by considering the minimization of

$$\ell(x(0), x(T)) + \Lambda(x),$$

where ℓ is an extended-valued lower semicontinuous function, as described in §1.3. In that case, the arc p can be chosen to satisfy the transversality condition[21]

$$(p(0), -p(T)) \in \partial \ell(x(0), x(T)).$$

The subject of necessary conditions for nonsmooth problems will be studied in depth in Chapter 4.

2.6. Hamiltonian growth conditions.

The global coercivity of Tonelli's Existence Theorem is a somewhat restrictive hypothesis, and so the need has been felt to replace it with subtler growth criteria. Whatever growth assumptions we impose should at least assure that for any arc x, the function $L(t, x(t), \dot{x}(t))$ majorizes an integrable function; otherwise, the very meaning of (P) is in doubt. Since $x(t)$ is bounded, whereas $\dot{x}(t)$ may not be, a natural way to assure this is to require that for any bounded subset S of $\mathbb{R}^n$, there exist a vector p and an integrable function $\varphi(t)$ on $[0, T]$ such that

$$L(t, x, v) \geq \langle p, v \rangle - \varphi(t), \quad t \in [0, T], \quad x \in S, \quad v \in \mathbb{R}^n.$$

We may rewrite this inequality as follows:

$$\sup_{v \in R^n} \{\langle p, v \rangle - L(t, x, v)\} \leq \varphi(t).$$

It so happens that the left side of this inequality is a celebrity. It is a function $H(t, x, p)$ called the *Hamiltonian* (corresponding to L), and it plays a major role in the following chapters.

Following Rockafellar [R1975], we call *basic growth condition* the requirement that for any bounded set S and for any p in $\mathbb{R}^n$, there is an integrable function φ such that

$$H(t, x, p) \leq \varphi(t), \quad t \in [0, T], \quad x \in S.$$

It is easy to verify that the basic growth condition is a weaker condition than coercivity (T2)′. Nonetheless it is sufficient, when supplemented by a unilateral state constraint, to yield the following existence theorem.

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[21] In fact, we can obtain the more precise condition $(p(0), -p(T)) \in \hat{\partial} \ell(x(0), x(T))$, a type of refinement first pointed out by Mordukhovich [M1980]. An argument to obtain this would first reduce to the case in which ℓ is Lipschitz, by a device used in [C1975b]. Then we have $0 \in \hat{\partial}(\ell + \Lambda)(x) \subseteq \hat{\partial} \ell(x) + \partial \Lambda(x)$, following which the characterization of $\partial \Lambda$ in [C1983, 2.7.4] leads to the result.

THEOREM 2.3 (ROCKAFELLAR). *If L satisfies* (T1)′ *and* (T3)′, *and if H satisfies the basic growth condition, then the following problem admits a solution whenever there are feasible arcs:*

$$\text{minimize}\,\{\Lambda(x):\ x \in AC,\ \|x\| \le R,\ x(0) = x_0,\ x(T) = x_T\}.$$

The addition of the state constraint $|x(t)| \le R$ is an element that makes a big difference. For example, the Euler equation is no longer a necessary condition for optimality (the reader is invited to ponder where the standard variational argument fails). Since the Euler equation is relatively simple, and is often itself the desired goal (as in §2.8), the most common step after applying the theorem is to manufacture an ad hoc argument to show that the solutions satisfy $\|x\| < R$, if R has been chosen judiciously. The reader is invited to show that an a priori bound of this type can be proven quite easily when global coercivity is present. For this reason, we can consider Theorem 2.3 to be a true generalization of Tonelli's Existence Theorem, despite the addition of the state constraint.

We shall not discuss the proof of Theorem 2.3, but we remark that like Tonelli's and all its other modern extensions, it is a *topological* theorem based on compactness of the level sets

$$\{x \in AC:\ \Lambda(x) \le \lambda,\ \|x\| \le R\}.$$

Recently a different ("intermediate") approach to existence was developed by Clarke and Loewen [CL1988], in which characteristics of the extremals (i.e., the behavior of the necessary conditions) are immediately taken into account when studying the existence issue itself, rather than subsequently, as is usually the case.

2.7. Intermediate existence and regularity.

We now discuss an approach to existence in which the hypotheses are more restrictive in some respects than those of Tonelli's Existence Theorem. However, when the hypotheses do apply, the conclusions are correspondingly stronger; specifically, existence is assured in the class of locally Lipschitz functions, whose central role we have spoken of earlier. Another advantage of this approach is that it allows the possibility of considering "slow-growth" (noncoercive) Lagrangians.

We now illustrate this "intermediate existence" theory with two examples of results in [CL1989]. We continue to suppose that the Lagrangian L is locally Lipschitz (i.e., (T1)′), and for ease of exposition we also suppose that it is strictly convex in v.

The behavior of extremals will be studied in terms of the following situation. We consider minimizing

$$\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt$$

over the smooth arcs x on $[t_0, t_1]$ that have prescribed endpoints and satisfy $|x(t)| < R$, $t \in [t_0, t_1]$, where $R > 0$ is given and $[t_0, t_1]$ is a subinterval of $[0, T]$. For further parameters r and τ, $0 < r < \tau$, we denote by $\Delta_R(r, \tau)$ the infimum of $t_1 - t_0$ over all such problems admitting a solution x such that

$$\min\,\{|\dot{x}(t)|:\ t_0 \le t \le t_1\} \le r, \qquad \max\ \{|\dot{x}(t)|:\ t_0 \le t \le t_1\} = \tau.$$

In other terms, $\Delta_R(r,\tau)$ measures how fast solutions x of certain bounded subproblems defined in terms of L can evolve from $|\dot{x}| \le r$ to $|\dot{x}| = \tau$. We say that L satisfies the *extremal growth condition* provided that for each R, for each r sufficiently large, we have

$$\lim_{\tau\to\infty} \Delta_R(r,\tau) = +\infty.$$

It turns out that the extremal growth condition plays an important, and somewhat unifying, role in obtaining existence and regularity results. It is a condition that can sometimes be verified on an ad hoc basis, and it can also be shown to hold automatically in certain contexts. In the following proposition, the phrase "boundedly coercive" means that for every $R > 0$, there is a function θ_R satisfying (T2)′ when s is restricted to $|s| \le R$.

PROPOSITION 2.2. *The extremal growth condition holds in each of the following circumstances:*

(i) *L is boundedly coercive and autonomous.*

(ii) *L is boundedly coercive and satisfies the generalized Tonelli–Morrey growth condition* (2.11).

(iii) *L is of the form $\varphi(s)(1+|v|^2)^{1/2}$, where φ is locally Lipschitz and bounded away from zero.*

(iv) *H is locally Lipschitz and satisfies the following two conditions for every $R > 0$:*

(a) $\lim_{|p|\to\infty} \inf_{\substack{|x|\le R \\ t\in[0,T]}} H(t,x,p) = \infty$

(b) *There is a nonnegative function $k_R \in L^1[0,T]$ such that $|\zeta| \le k_R(t)[1 + |H(t,x,p)|]$ for all $\zeta \in \partial_t H(t,x,p)$, whenever (t,x,p) is a point in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ having $|x| < R$.*

Proof. We shall merely sketch the proof of (ii). Let x solve the problem on $[t_0,t_1]$ described above in connection with the definition of $\Delta_R(r,\tau)$. The necessary conditions of Theorem 2.2 apply and affirm the existence of an arc p on $[t_0,t_1]$ satisfying $(\dot{p},p) \in \partial L(t,x,\dot{x})$. The Tonelli–Morrey condition (or rather, its extension to nonsmooth L) implies

$$|\dot{p}| \le c|L(t,x,\dot{x})| + k(t)|p| + m(t).$$

Because $|x|$ is bounded by R, and because $|\dot{x}(t)|$ is no greater than r at least once, this yields via Gronwall's lemma a bound of the form $|p(t)| \le \Phi_1(t,R,r)$, where Φ_1 is continuous. The coercivity (for $|x| \le R$) implies that $|L_v(t,x,v)|$ goes to $+\infty$ with $|v|$, so that we deduce a bound $|\dot{x}(t)| \le \Phi_2(t,R,r)$. Using this, we demonstrate that as τ goes to $+\infty$, $t_1 - t_0$ must become unbounded. The full details of the argument are given in [CL1989b], where it is shown that (2.11) may be replaced by the following extended condition: for every $R > 0$, there are nonnegative constants c_0, c_1 and a nonnegative function $m \in L^1[0,T]$ such that

$$|q| \le (1 + |p|)[m(t) + c_0|L(t,x,v)| + c_1|v|]$$

whenever $q \in \partial_x L(t,x,v)$, $p \in \partial_v L(t,x,v)$, $|x| < R$ for some point (t,x,v) in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$. □

The problem we consider now is the version of (P) in which the additional constraint $||x|| < R$ is imposed; we label it (P_R).

$$(P_R) \qquad \text{minimize } \Lambda(x) : \ x \in AC, \ ||x|| < R, \ x(0) = x_0, \ x(T) = x_T.$$

Note the *strict* inequality in the state constraint, assuring that any solution is "interior."

THEOREM 2.4. *Let L satisfy the extremal growth condition, and suppose that for some $\alpha > 0$, for all $t \in [0,T]$, and for all x in $\mathbb{R}^n$ satisfying $|x| < R$, we have*

$$|p| \leq \alpha \ \Rightarrow \ H(t,x,p) \leq \frac{(R-m)\alpha - \Lambda(\overline{x})}{T}, \tag{2.13}$$

where $\overline{x}$ is any arc admissible for (P_R), and where m is any number satisfying $\min\{|x_0|, |x_T|\} < m < R$. Then there is at least one solution to (P_R), and all solutions are Lipschitz.

The growth condition (2.13) is reminiscent of the basic growth condition of the previous section, but is imposed only for "intermediate" values of p. The conclusion yields existence for "intermediate" problems in which $|x|$ is bounded above, and asserts that the solutions lie in the "intermediate" class of Lipschitz functions. Theorem 2.4 illustrates the nature of the trade-off in the intermediate existence theory: less growth is required, but the behavior of the extremals is constrained, and the conclusion is one that incorporates regularity together with existence. In contrast to the topological existence theorems cited in previous sections, Theorem 2.4 applies to situations in which the following level sets *fail* to be compact or closed:

$$\{x \in AC : \Lambda(x) \leq \lambda, \ ||x|| < R\}, \qquad \{x \in AC : \ \Lambda(x) \leq \lambda, \ ||x|| \leq R\}.$$

Theorem 2.4 can be used to derive existence theorems "in the small" (satisfy (2.13) by making T small) and "in the large" (satisfy (2.13) by making R large); this is another sense in which the theorem is "intermediate."

The last case of Proposition 2.2 demonstrates that the extremal growth condition holds for certain *slow-growth* Lagrangians. Here is a new existence theorem covering some such cases.

THEOREM 2.5. *Let L be bounded below and satisfy the extremal growth condition as well as*

$$\lim_{|v| \to \infty} \inf_{\substack{t \in [0,T] \\ x \in \mathbb{R}^n}} \frac{L(t,x,v)}{|v|} > 0.$$

Then there exists at least one solution to (P), and all solutions are locally Lipschitz.

2.8. The dual action and periodic Hamiltonian trajectories.

The theory of boundary value problems has always been closely linked to the calculus of variations. We shall now illustrate in that context the use of the results of the previous sections, in particular intermediate existence theory. The problem is the celebrated one of finding periodic solutions of the following Hamiltonian system:

$$J\dot{z}(t) = \nabla H(z(t)), \tag{2.14}$$

where the state z lies in $\mathbb{R}^{2n}$, H is a given C^1 function, and J is the $2n \times 2n$ matrix

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

(I is the $n \times n$ identity matrix). Among the numerous issues that are of interest (see for example [MW1989]) is whether (2.14) admits a solution of period T, which is equivalent to requiring $z(0) = z(T)$.

When the Hamiltonian H is convex, which we assume in this section, a certain variational principle introduced by Clarke [C1978b], [C1979a] has proven useful in the study of such questions (see also §§3.5, 4.7). To state it, we need the notion of the conjugate function H^*, which is defined by

$$H^*(\zeta) = \sup_z \{\langle \zeta, z \rangle - H(z)\}.$$

Then H^* is also convex. The operation above applied to H^* gives H (i.e., $H^{**} = H$), and certain reciprocity relationships hold between H and H^*. Regarding growth, for example, we have

$$H(z) \geq \frac{\epsilon}{2}|z|^2 \quad \forall z \iff H^*(\zeta) \leq \frac{\epsilon^{-1}}{2}|\zeta|^2 \quad \forall \zeta. \tag{2.15}$$

Although H is C^1 by assumption, H^* is not necessarily differentiable. The gradient of H and subdifferential of H^* are inverses in the following sense:

$$\zeta = \nabla H(z) \iff z \in \partial H^*(\zeta). \tag{2.16}$$

The *dual action* is the variational integral

$$D(z) := \int_0^T \{\frac{1}{2}\langle Jz(t), \dot{z}(t)\rangle + H^*(J\dot{z}(t))\}\, dt. \tag{2.17}$$

Note that its Lagrangian is nonsmooth, convex in $\dot{z}$, but nonconvex overall (i.e., in $(z, \dot{z})$). The property that explains the dual action's relevance to the Hamiltonian system (2.14) is summarized in the following proposition.

PROPOSITION 2.3. *Suppose that the state z is an extremal of the dual action, i.e., satisfies the Euler inclusion* (2.12) *together with some arc p. Then for some constant c, the arc $\tilde{z} = z + c$ satisfies* (2.14) *on* $[0, T]$.

Proof. When the Lagrangian is that of the dual action, the Euler inclusion (2.12) becomes (exercise)

$$\dot{p} = -\tfrac{1}{2}J\dot{z}, \qquad Jp + \frac{z}{2} \in \partial H^*(J\dot{z}).$$

The first relation gives $Jp = \frac{1}{2}z + c$ for some constant c, so the second becomes $z + c \in \partial H^*(J\dot{z})$. Invoking (2.15), we get the fact that $\tilde{z} = z + c$ satisfies (2.14). □

In view of Proposition 2.3, we can produce Hamiltonian trajectories by generating extremals of the dual action. The most straightforward way to do this is to minimize $D(z)$. This calls for an existence theorem; following this, we hope that regularity and necessary conditions will apply to yield the desired extremal.

We now give an example of a context in which this program can be carried out quite simply.[22]

THEOREM 2.6. *Let H satisfy the growth condition*

$$\mu + \beta|z|^{1+\epsilon} \leq H(z) \leq \gamma + \delta|z|^{2-\epsilon} \quad \forall z$$

for certain constants μ, β, γ, δ, and ϵ, with β, δ and $\epsilon > 0$. Then there exists a T-periodic solution of (2.14).[23]

Proof. It follows from the lower bound on H that H^* is finite everywhere, and hence locally Lipschitz, as well as convex in the velocity. In fact, it follows from the differentiability of H that H^* is strictly convex, so that L is strictly convex in the velocity.

We now consider the problem of minimizing $D(z)$ subject to $z(0) = z(T) = 0$. The first order of business is to prove that a solution exists. Note that L is not coercive, so that Tonelli's Existence Theorem does not apply. Note also that Theorem 2.3 would not be of use, for our goal is to write the Euler equation and invoke Proposition 2.3, which would not be possible in the presence of the closed state constraint $|x| \leq R$. We turn therefore to Theorem 2.4, for the addition of an open constraint $|x| < R$ preserves the Euler equation as a valid necessary condition.

We first calculate the Hamiltonian $\overline{H}(z,q)$ corresponding to the dual action Lagrangian:

$$\begin{aligned}
\overline{H}(z,q) &= \max_{\dot{z}} \left\{ \langle q, \dot{z} \rangle - \tfrac{1}{2}\langle Jz, \dot{z} \rangle - H^*(J\dot{z}) \right\} \\
&= \max_{\dot{z}} \left\{ \langle Jq, J\dot{z} \rangle + \tfrac{1}{2}\langle z, J\dot{z} \rangle - H^*(J\dot{z}) \right\} \\
&= H\left(Jq + \frac{z}{2} \right) \qquad \text{(since } H^{**} = H\text{)}.
\end{aligned}$$

It follows from Proposition 2.2 (iv) that the extremal growth condition is present.

To invoke Theorem 2.4, we need only to verify that (2.13) holds for some R, α, m, $\overline{x}$. Let us take $\overline{x} \equiv 0$. Since $H^*(0) = \max\{-H(z) : z \in \mathbb{R}^{2n}\} \leq -\mu$, we have $\Lambda(\overline{x}) \leq -T\mu$. And if we have $|q| \leq \alpha$, $|z| \leq R$, then

$$\begin{aligned}
\overline{H}(z,q) = H\left(Jq + \frac{z}{2} \right) &\leq \gamma + \delta \left| Jq + \frac{z}{2} \right|^{2-\epsilon} \\
&\leq \gamma + \delta \left(\alpha + \frac{R}{2} \right)^{2-\epsilon}.
\end{aligned}$$

Combining these observations, and bearing in mind that m can be an arbitrarily small positive number, we see that (2.13) will hold if R and α are such that

$$\gamma + \delta \left(\alpha + \frac{R}{2} \right)^{2-\epsilon} < \frac{R\alpha + T\mu}{T}.$$

[22] This is drawn from [CE1980]. See, for example, [C1985], [E1986], and [MW1989] for references on the vast subject of periodic trajectories.

[23] It can be shown that the trajectory produced by Theorem 2.6 has true (minimal) period T.

Set $\alpha = R/2$; then the last inequality becomes

$$\gamma + \delta R^{2-\epsilon} < \frac{R^2}{2T} + \mu,$$

which is certainly valid for R sufficiently large.

Knowing now that (P_R) has a solution, we would like to write the Euler equation. Precisely because Theorem 2.4 asserts that the solution is Lipschitz, we may do this. The arc $\tilde{z}$ of Proposition 2.3 is now seen to be the required periodic trajectory. □

The point of the dual action is its unique feature: it has extremals that are Hamiltonian trajectories (modulo translation by a constant), while at the same time it is susceptible to minimization (and other critical point methods). This is in contrast to the classical action $\int_0^T \{\langle Jz, \dot{z}\rangle + H(z)\}\, dt$, which is highly indefinite. We shall see other uses of the dual action in a different context in Chapter 4 (§4.7).

2.9. Higher-order regularity.

The methods used to prove Theorem 2.1 have recently permitted, for the first time, a treatment of regularity for variational problems involving derivatives of more than the first order. Such problems have a long history, for example, in connection with beams and rods. We shall briefly describe such a result here, limiting ourselves to the second-order case for ease of presentation. Accordingly, consider the problem of minimizing

$$\int_0^T L(t, x(t), \dot{x}(t), \ddot{x}(t))\, dt$$

over a class X of functions satisfying prescribed boundary conditions at 0 and T. In this case, it is appropriate to prescribe both x and $\dot{x}$ at the endpoints and to use for X the class of functions x that are continuously differentiable and whose derivative $\dot{x}$ is absolutely continuous.

Tonelli's approach to existence carries over to the present context. It suffices to impose hypotheses analogous to (T1)′–(T3)′ (namely, that $L(t, s, v, w)$ be locally Lipschitz in (t, s, v, w), convex in w, and coercive in w) in order to deduce the existence of a solution to the problem. We now wish to discuss regularity properties of the solution, following recent work of Clarke and Vinter [CV1989c], [CV1989d].

We shall say that τ in $[0, T]$ is a *regular point* of x provided that we have

$$\liminf_{\substack{s,t \to \tau \\ 0 \le s \le \tau \le t \le T \\ s \ne t}} \left\{ \frac{|x(t) - x(s) - \dot{x}(s)(t-s)|}{|t-s|^2} + \frac{|\dot{x}(t) - \dot{x}(s)|}{|t-s|} \right\} < \infty.$$

It can be shown that all Lebesgue points of $\ddot{x}$ are regular and thus regular points have full measure in $[0, T]$.

THEOREM 2.7. *Let τ be a regular point of the solution x. Then there is a neighborhood of τ in $[0,T]$ in which $\ddot{x}$ is essentially bounded.*

As was the case with Theorem 2.1 involving a single derivative, a number of stronger conclusions follow routinely from Theorem 2.7 under various additional hypotheses. Let us note just one of these before pointing out a significant difference between Theorem 2.1 and Theorem 2.7.

COROLLARY 2.6. *Suppose that for a certain constant c, integrable function γ, and locally bounded function r, the Lagrangian L satisfies the growth condition*

$$|\partial_{s,v}L(t,s,v,w)| \leq c(|L(t,s,v,w)| + |w|) + \gamma(t)r(s,v).$$

Then the solution x is such that $\ddot{x}$ is bounded on $[0,T]$.

As before, a global conclusion such as this one justifies writing the necessary conditions in a strong form. The result compares closely with Corollary 2.4 , and here we also refer to the growth condition as that of Tonelli–Morrey.

In considering the higher-order extension of Corollary 2.2 (for autonomous problems), we become aware of what seems to be a significant distinction. To see this, consider the autonomous problem in which the Lagrangian is given by (for $n = 1$)

$$L(t,s,v,w) = |s^2 - v^5|^2\,|w|^{22} + \epsilon|w|^2.$$

Clearly this Lagrangian, which satisfies the hypotheses, is inspired by the Ball–Mizel problem of §2.3. It is possible to show that the function $x(t) = kt^{5/3}$ is an extremal for L (in a classical sense appropriate to the problem) for a certain constant k. Yet $\ddot{x}$ fails to be bounded, so this would show (if x indeed solves the problem) that, for higher-order problems, being autonomous does not entail regularity to the same extent. We emphasize, however, that the optimality of x has yet to be confirmed.

CHAPTER 3

Verification Functions and Dynamic Programming

In the preceding chapter we explored, for the basic problem in the calculus of variations, the principal issues pertinent to the deductive method: existence on the one hand, and the twin issues of regularity and necessary conditions on the other. In this chapter we study the technique of verification functions, which unifies all the main inductive methods. We will also see how this study leads to a complex of ideas around dynamic programming and the Hamilton–Jacobi equation. Once again value functions play a prominent role.

3.1. Verification functions.

Let us illustrate the basic idea with a simple example, that of minimizing

$$\Lambda(x) = \int_0^1 |\dot{x}(t)|^2 \, dt$$

over the arcs x (i.e., absolutely continuous functions) satisfying $x(0) = 0$, $x(1) = 1$. The solution to this problem by the deductive method requires the following steps:

(i) An absolutely continuous solution $\hat{x}$ to the problem exists by Tonelli's Existence Theorem.

(ii) Regularity theorems (e.g., Theorem 2.1, Corollary 2.2) imply that $\hat{x}$ has a bounded derivative.

(iii) Since $\hat{x}$ has a bounded derivative, the Euler equation in integral form applies. We deduce that $\hat{x}$ is C^2 and satisfies $d^2\hat{x}/dt^2 = 0$.

(vi) It follows that $\hat{x}(t) \equiv t$ is the unique solution.

This solution involves a lot of machinery. What if we wished to prove to a first-year undergraduate that $\hat{x}$ above is indeed the solution? Here is one way that recommends itself by its elementary nature. Observe the general inequality

$$v^2 \geq 2v - 1$$

and replace v by $\dot{x}(t)$, where x is admissible for the problem. Now integrate over $[0,1]$ to get

$$\Lambda(x) = \int_0^1 |\dot{x}(t)|^2\,dt \geq \int_0^1 \{2\dot{x}(t) - 1\}\,dt = 2[x(1) - x(0)] - 1 = 1.$$

But when x is $\hat{x}$, the above holds with equality; hence $\hat{x}$ minimizes Λ subject to the given boundary conditions. Generating and exploiting this kind of coincidence is our first order of business. (We could also invoke convexity in this example, depending on the student. Convexity is a special case of the technique on which we are focusing.)

Here is the verification method in general terms. Recall the problem (P): minimize

$$\Lambda(x) = \int_0^T L(t, x(t), \dot{x}(t))\,dt$$

over the arcs in AC satisfying $x(0) = x_0$, $x(T) = x_T$. Suppose that a suspect $\hat{x}$ is at hand, and we wish to confirm that $\hat{x}$ solves (P).

The basic idea is very simple. Suppose that a C^1 function $\varphi(t,s)$ exists such that, for $t \in [0,T]$ and $(s,v) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$L(t,s,v) \geq \varphi_t(t,s) + \varphi_s(t,s) \cdot v \tag{3.1}$$

and

$$\text{equality holds a.e. in (3.1) along } \hat{x}. \tag{3.2}$$

(This last phrase means that (3.1) holds with equality when $(t,s,v) = (t, \hat{x}(t), \hat{v}(t))$ for almost all $t \in [0,T]$, where $\hat{v}$ denotes $d/dt\,\hat{x}$.) Then the existence of such a φ, which we call a *verification function*, confirms the optimality of $\hat{x}$ for (P). For let x be any arc admissible for (P), and consider (3.1) with $(t,s,v) = (t, x(t), \dot{x}(t))$. Integrate over $[0,T]$ to get

$$\begin{aligned}\Lambda(x) = \int_0^T L(t, x(t), \dot{x}(t))\,dt &\geq \int_0^T \frac{d}{dt}\varphi(t, x(t))\,dt \\ &= \varphi(T, x_T) - \varphi(0, x_0).\end{aligned}$$

This last constant is therefore a lower bound on $\Lambda(x)$. But observe that if $x = \hat{x}$, then this lower bound is attained; hence $\hat{x}$ solves (P). In the example above, where $L(t,s,v) = v^2$, our argument corresponded to $\varphi(t,s) = 2s - t$.

The principal questions concerning this inductive method are evident. Does there always exist a verification function φ to confirm that $\hat{x}$ is optimal (when it is)? And how do we find verification functions?

The value function $V(\tau, y)$, defined as follows, provides a great deal of insight into both these issues:

$$V(\tau, y) = \inf \int_0^\tau L(t, x, \dot{x})\,dt, \quad x \in AC, \quad x(0) = x_0, \quad x(\tau) = y.$$

(We refer to the problem defined by the right side as $(P_{\tau,y})$.) Note that (τ, y) is simply the parameter specifying the horizon and endpoint constraint of a family of problems all of the same type as (P). Suppose now that $\hat{x}$ solves (P) and that

φ is a verification function for $\hat{x}$. Without loss of generality, suppose $\varphi(0, x_0) = 0$. Let x be any arc on $[0, \tau]$ satisfying $x(0) = x_0$, $x(\tau) = y$. Integrate the inequality (3.1) evaluated along x to obtain

$$\int_0^\tau L(t, x, \dot{x})\, dt \geq \varphi(\tau, y).$$

Taking the infimum over the x feasible for $(P_{\tau,y})$ gives

$$V(\tau, y) \geq \varphi(\tau, y). \tag{3.3}$$

Now, for each τ in $(0, T]$, it follows from the optimality of $\hat{x}$ for (P) that $\hat{x}$ solves $(P_{\tau,\hat{x}(\tau)})$. This inherited optimality for certain subproblems is sometimes called "the principle of optimality." Interpreted in terms of V, this fact reads as follows:

$$V(\tau, \hat{x}(\tau)) = \int_0^\tau L(t, \hat{x}, \hat{v})\, dt = \varphi(\tau, \hat{x}(\tau)), \qquad 0 < \tau \leq T. \tag{3.4}$$

In summary then, V majorizes φ, while agreeing with φ along $\hat{x}$. We have seen in Chapter 1 that value functions are not always differentiable. The possible relevance here to the verification technique is the following: if V is not differentiable, if V has a "downward corner" at a point $(\tau, y) = (\tau, \hat{x}(\tau))$ (a corner that has the nature of $t \longrightarrow -|t|$ at $t = 0$), then no C^1 function φ can agree with V at that point while it is majorized by V everywhere. Thus the existence of any such φ would be ruled out. On the other hand, if V happens to be smooth, our calculation suggests that φ could possibly be V itself. Of course, the very definition of V is somewhat difficult at $t = 0$; $V(0, x_0)$ seems naturally defined as 0, but $V(0, y)$ for $y \neq x_0$ should probably be assigned the value $+\infty$.

Our first concern turns out to be justified. There are very regular problems (P) (with smooth Lagrangians) that generate value functions V having the type of corner alluded to above.[24] Thus the solutions of these problems do not admit verification functions.

As for our second thought, let us now confirm that some optimism is justified.

PROPOSITION 3.1. *Let L be continuous, and suppose the infimum that defines $V(\tau, y)$ is attained for each (τ, y) in $(0, T] \times \mathbb{R}^n$. Suppose further that V is differentiable. Then V satisfies* (3.1), *and if $\hat{x}$ solves (P), then V satisfies* (3.2) *as well.*

Proof. The second assertion is an immediate consequence of differentiating (3.4). As for the first assertion, let x solve $(P_{\tau,y})$, and let any V be given. By considering the arc z that agrees with $x(t)$ for $t \leq \tau$ and that equals $x(\tau) + (t - \tau)v$ for $t > \tau$, we derive, for any $\Delta > 0$ small, by definition of V,

$$\begin{aligned} V(\tau + \Delta, y + \Delta v) &\leq \int_0^{\tau+\Delta} L(t, z, \dot{z})\, dt \\ &= \int_0^\tau L(t, x, \dot{x})\, dt + \int_\tau^{\tau+\Delta} L(t, x(\tau) + (t - \tau)v, v)\, dt. \end{aligned}$$

[24] Fleming [FL1969] establishes some smoothness properties of V under certain hypotheses.

The first term on the right coincides with $V(\tau, y)$. If we subtract it from both sides, divide by Δ, and let Δ tend to zero, we obtain precisely (3.1) (for $\varphi = V$). $\square$

The fact that V may be a verification function typically can be used as follows. Form a conjecture regarding the solutions of $(P_{\tau,y})$ for all (τ, y); then calculate V provisionally through substitution of the conjectured solutions. If then (3.1) and (3.2) hold for $\varphi = V$, the conjecture is verified. We thus wind up solving a whole family of problems rather than just (P) itself. Incidentally, the whole approach admits a local form where only (τ, y) near the values $(t, \hat{x}(t))$ are considered, but we do not pursue it here, since no additional insight results.

Let us illustrate this by an example, the one with which we began this section. The problem $(P_{\tau,y})$ is

$$\begin{aligned} &\text{minimize } \int_0^\tau \dot{x}^2\, dt \\ &\text{subject to } x(0) = 0, x(\tau) = y. \end{aligned}$$

We suspect the solution to be the arc $x(t) = yt/\tau$, for which the integral in question turns out to be y^2/τ. We verify easily that (3.1) and (3.2) hold (note that we had proposed a different verification function before). This almost completes the proof that $\hat{x}(t) \equiv t$ solves (P), except for the problem of dealing with the bad behavior of y^2/τ at $\tau = 0$. If the argument proving the sufficiency of (3.1) and (3.2) is modified so that we integrate only over $[\epsilon, 1]$ for $\epsilon > 0$, it yields

$$\int_\epsilon^1 \dot{x}^2\, dt \geq 1 - \frac{x(\epsilon)^2}{\epsilon}$$

for any admissible arc x. In the limit, provided that $x(\epsilon)^2/\epsilon$ tends to zero, this inequality gives the required conclusion. But since we know solutions of $(P_{\tau,y})$ to be Lipschitz by the regularity results of Chapter 2, we may restrict ourselves to this class, for which $x(\epsilon)^2/\epsilon$ does indeed tend to zero. (Note how regularity remains a pervasive issue even in this context.)

We could have avoided the difficulty with V at $\tau = 0$ in the above example another way by "backing up" the problem to one on $[-\epsilon, 1]$ for some $\epsilon > 0$. Consider the problem

$$\begin{aligned} &\text{minimize } \int_{-\epsilon}^\tau \dot{x}^2\, dt \\ &\text{subject to } x(-\epsilon) = -\epsilon, x(\tau) = y. \end{aligned}$$

Again we suspect linear arcs. If we calculate the value function $\widetilde{V}(\tau, y)$ on the basis of this conjecture, we get

$$\widetilde{V}(\tau, y) = \frac{(y + \epsilon)^2}{\tau + \epsilon}.$$

This function satisfies (3.1) and (3.2) for the new extended interval (for $\hat{x}(t) \equiv t$ extended to $[-\epsilon, 1]$), and all the more for the original one. Further, observe that $\widetilde{V}$ presents no difficulties at $\tau = 0$.

This backing-up technique lies at the heart of the proof of the classical sufficiency theorem in the calculus of variations, although it is not always phrased this way. A given extremal $\hat{x}$ on $[0,T]$ (say) is extended to one on $[-\epsilon,T]$. A field of extremals all passing through $(-\epsilon,\hat{x}(-\epsilon))$ is shown to cover a neighborhood of the graph of $\hat{x}$ on $[0,T]$. Then $\widetilde{V}(\tau,y)$, defined as the integral over $[-\epsilon,\tau]$ along the (unique) element of the field passing through (τ,y), is a smooth function (for $\tau \geq 0$), satisfying (3.1) and (3.2). This interpretation of the classical method of fields, the view that the verification function method is a unifying one for both weak and strong sufficiency, and the improvement over classical results that can be obtained, are discussed in detail in [CZ1986].

We have seen that a verification function can sometimes be produced by calculating an appropriate value function. Note that there is no uniqueness of verification functions. In the example above, an entire family of suitable ones (parametrized by ϵ) was found. Note also that the one we gave initially in discussing the example ($\varphi(t,s) = 2s-t$) is unrelated to the value function connected with the problem. In fact, it was simply produced in an ad hoc trial-and-error fashion which, although not a systematic method like value functions or field theory (or the Hamilton–Jacobi equation to be discussed presently), remains an occasionally useful way to find verification functions. A nontrivial example of this will be given below in connection with the Ball–Mizel problem defined in Chapter 2.

We have mentioned the unifying role played by the method of verification functions without discussing sufficiency results based upon convexity. When $L(t,\cdot,\cdot)$ is convex for each t, which is considered the "convex case" of (P), an arc $\hat{x}$ solves (P) if and only if it admits an arc p satisfying the Euler inclusion (see §2.5):

$$(\dot{p},p) \in \partial_{x,v} L(t,\hat{x},\hat{v}) \quad \text{a.e.}$$

Now if this relation holds, it follows from convexity that we have

$$L(t,s,v) - L(t,\hat{x},\hat{v}) \geq \langle \dot{p}, s-\hat{x}\rangle + \langle p, v-\hat{v}\rangle.$$

If we proceed to define

$$\varphi(t,s) = \langle p(t), s-\hat{x}(t)\rangle + \int_0^t L(\tau,\hat{x},\hat{v})\,d\tau,$$

then this last inequality is equivalent to (3.1), thus demonstrating that φ is a verification function for $\hat{x}$, since (3.2) also is easily verified. Thus the convex case is subsumed by the verification method as well.

3.2. The Hamiltonian.

It is time for an important addition to our cast of characters. The *Hamiltonian* corresponding to the Lagrangian L is the function $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$H(t,s,p) = \max_{v \in R^n} \{\langle p,v\rangle - L(t,s,v)\}. \tag{3.5}$$

We have seen in Chapter 2 how H arises in a natural way in connection with growth conditions on L. Note that under the continuity and coercivity hypotheses on L that were present in the preceding chapter, which we posit now as well, the fact that the maximum is attained (and that H is finite) follows automatically. In the next chapter we shall discuss the historical antecedents of H, as well as its role in necessary conditions. For the moment we wish only to note its use in the following, in which $V(\tau, y)$ continues to denote the value of $(P_{\tau,y})$.

PROPOSITION 3.2. *Let the hypotheses of Proposition* 3.1 *hold, and assume in addition that V is C^1 and H is continuous. Then V satisfies the Hamilton–Jacobi equation*

$$V_t(t,x) + H(t,x,V_x(t,x)) = 0. \tag{3.6}$$

Proof. If in (3.1) (for $\varphi = V$) we transpose the Lagrangian to the right side and take the maximum over v, we obtain

$$V_t(t,x) + H(t,x,V_x(t,x)) \leq 0. \tag{3.7}$$

Now let x^* on the interval $[0,t]$ be an arc solving $(P_{t,x})$ (x^* exists by hypothesis); then equality holds in (3.1) along x^* (by Proposition 3.1). Thus for any r in $(0,t)$ such that $\dot{x}^*(r)$ exists, we have

$$V_t(r,x^*(r)) + \max_v\{\langle V_x(r,x^*(r)),v\rangle - L(r,x^*(r),v)\} = 0,$$

i.e., equality in (3.7) for (t,x) replaced by $(r,x^*(r))$. Taking limits along a sequence of such points r converging to t shows that equality holds in (3.7). □

The Hamilton–Jacobi equation plays a central role in the calculus of variations, related as it is to such far-reaching issues as extremal fields, canonical transformations, invariance, wave propagation, and, as we now see, value functions. Its connection with our present concern, inductive methods for (P), is demonstrated in the following proposition.

PROPOSITION 3.3. *Let φ be any C^1 solution of the Hamilton–Jacobi equation, and let $\hat{x}$ be an arc admissible for (P) such that, for almost all t in $[0,T]$,*

$$\max_v\{\langle \varphi_x(t,\hat{x}(t)),v\rangle - L(t,\hat{x}(t),v)\} \tag{3.8}$$

is attained at $v = \hat{v}(t)$, where $\hat{v}(t)$ signifies $d/dt\,\hat{x}(t)$. Then $\hat{x}$ solves (P).

The proof is simple, and we leave it as an exercise for the reader. Indeed, it is virtually the same as the proof given that (3.1) and (3.2) imply the optimality of $\hat{x}$.

We see that the Hamilton–Jacobi equation is one potential way to produce a verification function. We remark that verification functions do not have to be solutions of the equation (note that the set of verification functions for $\hat{x}$ is convex, while the solutions to the equation do not form a convex set). Nonetheless, the last proposition is very significant in one way: it *begins* with a function φ not bearing any relation to any arc $\hat{x}$ and *then* speaks of an arc $\hat{x}$ satisfying (3.8). This hints at a possibly larger role for the Hamilton–Jacobi equation than a purely inductive one. Perhaps, together with (3.8), it could be the basis for the computation of optimal arcs. We shall see later how this idea, when discretized, leads directly to the technique known as *dynamic programming.*

An entire approach to dynamic optimization bears this name. It has proceeded in the continuous case by assuming smoothness of the value function, an assumption that is not only unverifiable in practice but demonstrably unsatisfied even in the simplest cases. Still, the approach has heuristic value and computational worth when discretized and can be made rigorous in the context of nonsmooth analysis, as will be seen.

The following proposition demonstrates how, under strong smoothness hypotheses, the dynamic programming approach uses the value function to obtain necessary conditions for optimality, in this case, the Euler equation.

PROPOSITION 3.4. *Let the value function V, the Lagrangian L, and the Hamiltonian H be C^2, and suppose that $\hat{x}$ solves (P). Define the arc p via*

$$p(t) = V_x(t, \hat{x}(t)). \tag{3.9}$$

Then p satisfies

$$\dot{p}(t) = \nabla_x L(t, \hat{x}(t), \hat{v}(t)) \quad a.e., \tag{3.10a}$$

$$p(t) = \nabla_v L(t, \hat{x}(t), \hat{v}(t)) \quad a.e. \tag{3.10b}$$

Proof. For any point p, the set of points v at which the maximum of the concave function $\langle p, v\rangle - L(t, x, v)$ is achieved is that satisfying $p = L_v(t, x, v)$. A consequence of the theory of conjugate convex functions is that there is precisely one such v, namely, $v = H_p(t, x, p)$. It follows from the Hamilton–Jacobi equation that when $(t, x, p) = (t, \hat{x}(t), p(t))$, the associated v is given almost everywhere by $\hat{v}(t)$, and from this that $p(t)$ coincides almost everywhere with $L_v(t, \hat{x}(t), \hat{v}(t))$. Observe also that we have the formula

$$H(t, x, p) = \langle p, H_p(t, x, p)\rangle - L(t, x, H_p(t, x, p)). \tag{3.11}$$

In view of this, the Hamilton–Jacobi equation (Proposition 3.2) may be written

$$V_t(t, x) + \langle V_x(t, x), H_p\rangle - L(t, x, H_p) = 0, \tag{3.12}$$

where H_p is evaluated at $(t, x, V_x(t, x))$. Differentiating the identity (3.12) with respect to x leads to

$$V_{tx}(t, x) + V_{xx}(t, x)H_p - L_x(t, x, H_p) + \{V_x(t, x) - L_v(t, x, H_p)\}\frac{\partial}{\partial x}H_p = 0.$$

Now evaluate this along $\hat{x}$. Note that then, as shown above, the term in braces vanishes and H_p coincides almost everywhere with $\hat{v}(t)$. We deduce

$$V_{tx}(t, \hat{x}(t)) + V_{xx}(t, \hat{x}(t))\hat{v}(t) - L_x(t, \hat{x}(t), \hat{v}(t)) = 0.$$

The first two terms clearly coincide with $d/dt\, p(t)$ almost everywhere, and so the proof of Proposition 3.4 is complete. □

Note that (3.10) implies the Euler equation (Chapter 2):

$$\frac{d}{dt}L_v(t, \hat{x}(t), \hat{v}(t)) = L_x(t, \hat{x}(t), \hat{v}(t)) \quad \text{a.e.}$$

What is more, it gives an interpretation of $p(t)$ (the "adjoint variable" of optimal control, or the "generalized momentum" of classical mechanics) as the marginal

rate of change of the value function V in the state variable. We also remark that differentiating (3.11) with respect to x and then substituting $(t, \hat{x}(t), p(t))$ proves that we have

$$-\frac{d}{dt}p(t) = H_x(t, \hat{x}(t), p(t)) \quad \text{a.e.,} \tag{3.13a}$$

$$\frac{d}{dt}\hat{x}(t) = H_p(t, \hat{x}(t), p(t)) \quad \text{a.e.} \tag{3.13b}$$

This is known as the canonical form of the Euler equation. In Chapter 2 (§2.8), we saw such Hamiltonian systems in a different guise. Simply set $(x, p) = z$ to write (3.13a) and (3.13b) in the form $J\dot{z} = \nabla_z H(t, z)$.

In the next sections we shall show how the methods of nonsmooth analysis can free the verification function technique and the dynamic programming approach from unrealistic smoothness requirements.

3.3. Nonsmooth verification functions.

We impose in this section the hypotheses (T1)′–(T3)′ of the (extended) Tonelli Existence Theorem (§2.3), and we suppose that L is autonomous as well. As mentioned earlier, even smooth solutions of smooth variational problems satisfying these hypotheses may not admit C^1 verification functions. One way to react to this disappointing fact is to extend the method by allowing verification functions that are not smooth. This will also make it more likely that we will be able to use value functions themselves as verification functions. Let us now define parallels of (3.1) and (3.2) with subgradients instead of derivatives of φ. We shall call φ a (generalized) verification function for $\hat{x}$, provided that φ is locally Lipschitz and satisfies, for all (t, s, v),

$$\max\{\alpha + \beta \cdot v - L(t, s, v) : \ (\alpha, \beta) \in \partial\varphi(t, s)\} \leq 0, \tag{3.14}$$

$$\varphi(T, x_T) - \varphi(0, x_0) = \Lambda(\hat{x}). \tag{3.15}$$

It is easy to see that (3.14) and (3.15) are equivalent to (3.1) and (3.2) when φ is smooth, so we have a true generalization.

We claim that the existence of a generalized verification function φ for $\hat{x}$ implies that $\hat{x}$ solves (P). To prove this, it suffices to show that for any admissible arc x, the function

$$t \longrightarrow \varphi(t, x(t)) - \int_0^t L(s, x, \dot{x})\, ds \tag{3.16}$$

is nonincreasing; then

$$\varphi(T, x_T) - \varphi(0, x_0) \leq \int_0^T L(s, x, \dot{x})\, ds$$

with equality when $x = \hat{x}$ (in view of (3.15)). Now the function in (3.16) is absolutely continuous, and for any t at which it is differentiable and for which $\dot{x}(t)$ exists, it can be shown that its derivative is of the form

$$\alpha + \beta \cdot v - L(t, x(t), \dot{x}(t))$$

for some (α, β) in $\partial\varphi(t, x(t))$, provided in, addition, that t is a Lebesgue point of the integrand in (3.16); almost all t satisfy these three conditions. This expression is nonpositive (by (3.14)), which completes the proof.

The following result confirms that our extension of the verification function technique suffices to make it applicable in theory in every case.

THEOREM 3.1. *An admissible $\hat{x}$ solves (P) if and only if there is a generalized verification function for $\hat{x}$.*

Proof. The sufficiency was proven above. We now sketch the proof of necessity. The first step is to show the existence of a constant K such that $\hat{x}$ solves globally the free endpoint problem

$$\begin{aligned} &\text{minimize } K|x(0) - x_0| + \Lambda(x) \\ &\text{subject to } x(T) = x_T. \end{aligned}$$

This uses locally uniform upper estimates for $||\dot{x}||$, where x solves slightly perturbed versions of (P), estimates proved with the help of Theorem 2.1. Next we define $\varphi(\tau, y)$ as the minimum of

$$K|x(0) - x_0| + \int_0^\tau L(t, x, \dot{x})\, dt$$

over the arcs x satisfying $x(\tau) = y$. Note that $\varphi(0, \cdot) \equiv 0$, and that $\varphi(T, x_T)$ equals $\Lambda(\hat{x})$, whence (3.15). The upper estimates alluded to above are then used to prove that φ is locally Lipschitz. Whenever φ is differentiable (which is almost everywhere, by Rademacher's theorem), the argument given in the proof of Proposition 3.1 establishes inequality (3.1). The characterization of $\partial\varphi$ in terms of limits of $\nabla\varphi$ (§1.3, (1.7))) then shows that (14) holds. □

Note that by taking the supremum over v in (3.14), we can express it in the following equivalent way:

$$\max\{\alpha + H(t, x, \beta) : \ (\alpha, \beta) \in \partial\varphi(t, x)\} \leq 0.$$

The function φ in the proof of the theorem can be shown [C1982b] to satisfy a stronger condition than (3.14) — the following generalized Hamilton–Jacobi equation:[25]

$$\max\{\alpha + H(t, x, \beta) : \ (\alpha, \beta) \in \partial\varphi(t, x)\} = 0. \tag{3.17}$$

This reduces to the classical Hamilton–Jacobi equation if φ is C^1. We summarize in the following corollary.

COROLLARY 3.1. *An admissible arc $\hat{x}$ solves (P) if and only if there exists a locally Lipschitz solution φ of the generalized Hamilton–Jacobi equation satisfying* (3.15).

The verification function method can be useful even with functions that are not locally Lipschitz and without necessarily being limited to solutions of the

[25] Had we not already exceeded our quota of neologisms, we might have termed this a "subequation." Many generalizations of the Hamilton–Jacobi equation tailored for different purposes exist, such as "viscosity solutions"; see [CS1989], [F1989], and [VW1989] for relationships between this notion and ours, and other results.

Hamilton–Jacobi equation. A significant example of this arises in connection with the Ball–Mizel problem defined in §2.3. It can be shown [CV1984] that (for $\hat{x}$ the given candidate arc), the following function φ, which is not Lipschitz, satisfies (3.1) and (3.2) almost everywhere:

$$\varphi(t,s) = \int_0^t L(\tau,\hat{x},\hat{v})d\tau + ct^{-1/3}(s-\hat{x}(t)) - \lambda \max\,[\epsilon t^{1/3} - st^{-1/3}, 0],$$

where c, λ, and ϵ are certain constants. The first term of φ is a natural one to begin with in attempting to construct a verification function. The second term is also; it corresponds to $\dot{p}(t)(s-\hat{x}(t))$, where p has already been introduced above (see the closing remarks of §3.1). The last term involves a "corner"; it was found by trial and error after determining that no smooth terms involving s linearly or quadratically would do.

We have seen that nonsmooth analysis allows us to complete the theory of verification functions in a satisfactory way. We now consider its application to the dynamic programming approach to necessary conditions.

3.4. Dynamic programming and nonsmooth analysis.

It does not seem possible to retain all the charms of the arguments used to prove Proposition 3.4, since the function V in question is so far from being C^2 that it even fails at times to be C^1. We *can* hope to retain an interpretation in terms of marginal value changes of a multiplier p satisfying an appropriate extended necessary condition.

We now give such a result for (P), one that should be compared with Proposition 3.4. The proof exploits a novel auxiliary problem introduced in [CV1987]. The hypotheses on L remain those of the preceding section.

THEOREM 3.2. *Let $\hat{x}$ solve (P). Then there exists an arc p satisfying*

$$\begin{aligned} &p(t) \in \partial_x V(t,\hat{x}(t)) \quad \text{a.e.},\\ &\dot{p}(t) \in \partial_x L(\hat{x}(t),\hat{v}(t)) \quad \text{a.e.},\\ &p(t) \in \partial_v L(\hat{x}(t),\hat{v}(t)) \quad \text{a.e.} \end{aligned} \tag{3.18}$$

Proof (sketch). In a first step similar to that of the proof of Theorem 3.1, we find $K > 0$ such that (locally, for $\tau > 0$) $V(\tau, y)$ agrees with the minimum of

$$K|y - x(\tau)| + \int_0^\tau L(x,\dot{x})\,dt$$

over arcs x near $\hat{x}$ satisfying $x(0) = x_0$. Because solutions are Lipschitz (Theorem 2.1, Corollary 2.2), this function is Lipschitz, so it has a locally bounded subdifferential. For $\epsilon > 0$, we define

$$\begin{aligned} G_\epsilon(t) &:= \text{co}\,\{\zeta : \zeta \in \partial_x V(t,y),\ |y - \hat{x}(t)| \le \epsilon\},\\ \sigma_\epsilon(t,v) &:= \max\,\{\langle v,\zeta\rangle : \zeta \in G_\epsilon(t)\}. \end{aligned}$$

Suppose now that x is an arc satisfying $|x(t) - \hat{x}(t)| \le \epsilon$, $t \in [0,T]$. Arguing much as in the proof of Proposition 3.1, but in subdifferential terms, we can

establish that for $v(T)$ any measurable function on $[0,T]$ with values in the unit ball of $\mathbb{R}^n$, we have

$$\frac{d}{dt}V(t,x(t)) \le L(x(t),\dot{x}(t)-v(t)) + \sigma_\epsilon(t,v(t)).$$

By integrating we can obtain

$$\int_0^T \{L(x(t),\dot{x}(t)-v(t)) + \sigma_\epsilon(t,v(t))\}\,dt \ge V(T,x(T)) - V(0,x_0).$$

Equality holds in this relation when we take $v \equiv 0$, $x = \hat{x}$. This shows that $\hat{x}$ solves a certain auxiliary problem, one involving another state variable represented explicitly only by its derivative v. We may apply the necessary conditions [C1983, Thm. 4.4.3] to this nonsmooth problem. Following a limiting process in which $\epsilon \downarrow 0$, we get precisely the required conclusions. □

Remark. The term $V(T,x(T))$ can be transferred to the left side of the inequality in which it appears so as to become an explicit component of the auxiliary problem (in which $x(T)$ is now considered unconstrained). The resulting transversality condition establishes that (3.18) holds for $t = T$. A similar argument captures $t = 0$. Finally let us observe that in this autonomous setting, a standard device called the *Erdmann transform* can be invoked to replace (3.18) by a conclusion of the form

$$(3.18') \qquad (-h,p(t)) \in \partial V(t,\hat{x}(t)),$$

where h is a constant such that $H(x(t),p(t)) = h$ $(t \in [0,T])$, and where ∂V signifies the subdifferential with respect to (t,x). Note that now (3.18′) can be asserted for *all* t in $[0,T]$ since ∂V, in contrast to $\partial_x V$, is upper semicontinuous in (t,x).

3.5. The value function for the dual action.

In §2.8 we studied the special case of (P) in which Λ is the dual action functional

$$\Lambda(z) = \int_0^T \left\{\frac{1}{2}\langle Jz,\dot{z}\rangle + H^*(J\dot{z})\right\} dt$$

and the boundary constraints are $z(0) = z(T) = 0$ (z lies in R^{2n}). We showed that if z is a solution of the problem, then some translate of z by a constant defines a T-periodic solution of the Hamiltonian system defined by the convex function H. When $V(\tau,y)$ is the value function as defined in the preceding sections, it is possible to establish a relationship between the subdifferential $\partial V(T,0)$ and those T-periodic trajectories [C1987b]. We denote by Π the set of T-periodic trajectories whose translates are solutions to the problem (P) above. Any such trajectory z is such that $H(z(t))$ has a constant value h_z, called the energy. Let $\widetilde{J}$ denote the invertible $(2n+1)\times(2n+1)$ matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{bmatrix}$$

THEOREM 3.3. *The function $-V$ is locally Lipschitz and regular in the sense of* §1.4, *and we have*

$$\widetilde{J}\partial V(T,0) = \operatorname{co}\{[h_z, z(t)] : \ z \in \Pi, \ 0 \le t \le T\}.$$

It follows that V is not differentiable at $(T,0)$. This may be an advantage rather than a difficulty. Indeed, the theorem suggests that it would be useful to know $\partial V(T,0)$ as a means of computing or locating periodic solutions. The subdifferential of V is characterized by its one-sided directional derivatives in the classical sense (since $-V$ is regular). These derivatives can be estimated in principle, since the function values of V are readily calculated by well-known techniques such as Rayleigh–Ritz. In the case when (P) has a unique solution, the formula shows that $\widetilde{J}\partial V(T,0)$ would coincide with $\{h_z\} \times \operatorname{co}\{z(t) : \ 0 \le t \le T\}$, where z is a T-periodic trajectory. The extreme points of the set $\operatorname{co}\{z(t) : \ 0 \le t \le T\}$ are precisely the ones lying on the orbit $z(\cdot)$. Even in the general case of multiple solutions, we have at least partial information, as shown in the following corollary.

COROLLARY 3.2. *Let $[h,\zeta]$ be any extreme point of $\widetilde{J}\partial V(T,0)$. Then there exists a T-periodic trajectory that passes through the point ζ and has energy h.*

The function V satisfies a "dual Hamilton–Jacobi equation," which is the Hamilton–Jacobi equation corresponding to the dual action. Since V is nonsmooth, this is true in a generalized sense involving subdifferentials, as in §3.3. The smooth version of the equation (exercise) is

$$V_t(t,z) + H\left(J V_z(t,z) + \frac{z}{2}\right) = 0.$$

This equation subsumes the usual Hamilton–Jacobi equation, for if we seek solutions $V(t,x,p)$ of the form $-\frac{1}{2}\langle x,p\rangle + \varphi(t,x)$, then we find that φ must satisfy

$$\varphi_t(t,x) + H(x, \varphi_x(t,x)) = 0,$$

that is, the usual Hamilton–Jacobi equation.

3.6. The dynamic programming algorithm.

Let us examine Proposition 3.3 once more to see how (3.8) might be used to compute the optimal arc $\hat{x}$. The set of points v at which the maximum with respect to v of the expression $p \cdot v - L(t,x,v)$ is attained consists of precisely those v that satisfy $p \in \partial_v L(t,x,v)$, since the expression in question is concave. This last inclusion is equivalent to $v \in \partial_p H(t,x,p)$, by the theory of conjugate convex functions. We see, therefore, that (3.8) is equivalent to the following differential inclusion:

$$\dot{x}(t) \in \partial_p H(t, x(t), \varphi_x(t, x(t))) \quad \text{a.e.} \tag{3.19}$$

Thus if φ is a C^1 solution of the Hamilton–Jacobi equation, trajectories x of (3.19) are optimal, provided, of course, that they are admissible. The "correct" solution φ would have to be chosen (for example, the value function V, which raises the issue of nonsmoothness). And then there remains the problem of

selecting appropriately from the set on the right side of (3.19) and whether the ensuing trajectories x would be defined throughout $[0, T]$.

A lot of work has been done and still remains to be done on these major difficulties, particularly in a control setting, where the goal of an optimal "feedback" law such as (3.19) is so attractive. One way in which these difficulties can be addressed simultaneously involves passing to a discretized problem. Let us see how this might be carried out for the problem (P). We shall discretize in the t variable only.

For m a positive integer, set $\Delta = T/m$ and define $t_i = i\Delta$ $(i = 0, 1, \cdots, m)$. Let us define the *cost* of the transition from (t_i, x) to (t_{i+1}, y) as

$$C(t_i, x; t_{i+1}, y) := L\left(t_i, x, \frac{y - x}{\Delta}\right)\Delta.$$

We note that this approximates

$$\int_{t_i}^{t_{i+1}} L(t, z(t), \dot{z}(t))\, dt$$

if z is the linear arc through (t_i, x) and (t_{i+1}, y). The cost of a sequence of transitions

$$(t_i, x_i) \longrightarrow (t_{i+1}, x_{i+1}) \longrightarrow \cdots \longrightarrow (t_{i+r}, x_{i+r})$$

is defined as the sum of the r individual transition costs. Now consider the problem of finding the least costly sequence of m transitions beginning at $(t_0, x_0) = (0, x_0)$ and ending at $(t_m, x_T) = (T, x_T)$. It is reasonable to view this as a discretized (in t) version of the basic problem (P) and to conjecture that its value converges to that of (P) as Δ goes to 0. (We expect this generally to be the case, but we are merely postulating it in this informal discussion.)

Let us examine the dynamic programming algorithm for this discrete problem. We define $V(\tau, y)$, where τ is one of the points t_i for $i > 0$, as the least cost associated with a sequence of transitions from (t_0, x_0) to (τ, y). The central point is that V satisfies a sort of recursion, namely,

$$V(t_{i+1}, y) = \min_x \{V(t_i, x) + C(t_i, x; t_{i+1}, y)\}. \tag{3.20}$$

This merely says that along an optimal sequence, the state x associated to the prior stage $t = t_i$ is such that

$$V(t_{i+1}, y) = V(t_i, x) + C(t_i, x; t_{i+1}, y),$$

while for other values of x the right side of the equation can be no less than the left, by definition of V.

The relation (3.20) is called the Bellman equation, and it is used to solve the problem as follows. First observe that the function $V(t_1, x)$ is known: it is simply $C(t_0, x_0; t_1, x)$. Then use (3.20) to calculate $V(t_2, y)$ and so on until $V(t_m, \cdot)$ has been reached. The minimum cost is then $V(t_m, T)$. The $m - 1$ intermediate values of x that yield the successive minima in (3.20) determine the optimal transition path.

Let us subtract $V(t_i, y)$ from each side of (3.20), divide across by Δ, and write the dummy variable x in the form $y - \Delta v$, where v is another dummy variable. Bearing in mind how C is defined, we obtain

$$\frac{V(t_{i+1}, y) - V(t_i, y)}{\Delta} = \min_v \left\{ \frac{V(t_i, y - \Delta v) - V(t_i, y)}{\Delta} + L(t_i, y - \Delta v, v) \right\}.$$

If we could "pass to the limit" as $\Delta \to 0$ in this expression, we would arrive at

$$V_t(t, y) = \min_v \{-V_x(t, y) \cdot v + L(t, y, v)\},$$

which is the same as

$$V_t(t, y) + \max_v \{V_x(t, y) \cdot v + L(t, y, v)\} = 0,$$

which we recognize as the Hamilton–Jacobi equation for V. Thus the recursion (3.20) can be viewed as a discretization of the Hamilton–Jacobi equation.

3.7. The verification method for isoperimetric problems.

The method of verification functions, as well as the closely linked topics of value functions and the dynamic programming approach, are rather robust in adapting to a variety of situations beyond that of the basic problem (P), in particular those incorporating additional constraints. We illustrate this now for the isoperimetric problem

$$\text{minimize} \int_0^T L(t, x, \dot{x})\, dt$$

$$\text{subject to} \int_0^T G(t, x, \dot{x})\, dt = 0$$

and subject to the same endpoint constraints as (P).

Suppose that $\hat{x}$ is a candidate for optimality. A verification function for $\hat{x}$ is now defined to be a function $\varphi(t, x, y)$ satisfying

$$L(t, x, v) \geq \varphi_t(t, x, y) + \varphi_x(t, x, y) \cdot v + \varphi_y(t, x, y)\, G(t, x, v) \quad \text{for all } (t, x, y, v) \tag{3.21}$$

with equality when (t, x, y, v) is equal to

$$(t, \hat{x}(t), \int_0^t G(s, \hat{x}, \hat{v})\, ds, \hat{v}(t)).$$

The existence of φ confirms that $\hat{x}$ is a solution of the problem. Let x be any admissible arc, and consider (3.21) with the role of (t, x, y, v) played by

$$\left(t, x(t), \int_0^t G(s, x, \dot{x})\, ds, \dot{x}(t)\right).$$

Then the right side is the time-derivative of $\varphi(t, x(t), \int_0^t G(s, x, \dot{x})\, ds)$. Integrating across (3.21) from 0 to T yields

$$\int_0^T L(t, x, \dot{x})\, dt \geq \varphi(T, x_T, 0) - \varphi(0, x_0, 0)$$

with equality when $x = \hat{x}$.

The same issues arise in this context as in (P), notably how to find a function φ satisfying (3.21). We leave as an exercise the following extension of Proposition 3.1.

PROPOSITION 3.5. *Suppose that for each (τ, y, r) the following problem has a solution:*

$$\text{minimize } \int_0^\tau L(t, x, \dot{x})\, dt$$
$$\text{subject to } \int_0^\tau G(t, x, \dot{x})\, dt = r$$

where the boundary conditions are $x(\tau) = y$, $x(0) = x_0$. Let the value be denoted $V(\tau, y, r)$, and suppose that V is differentiable. Then V satisfies (3.21), *and equality holds along any optimal arc.*

Example. Let us illustrate the use of the technique for isoperimetric problems by an application to inequalities.[26] As is so often the case, the basic idea will have to undergo some ad hoc modifications to deal with nonsmoothness and with some special features of the problem. The *logarithmic Sobolev inequality* is an important one in probability and mathematical physics. It can be reduced to show that for any positive-valued function x on $[0, \infty)$ satisfying

$$\int_0^\infty (x(t))^2\, dt = \frac{\sqrt{\pi}}{2},$$

we have

$$\int_0^\infty \left\{ \frac{1}{2}(\dot{x}(t))^2 - (x(t))^2 \ln |x(t)| \right\} dt \le \frac{\sqrt{\pi}}{4}.$$

This problem features an infinite horizon and free endpoints, but is amenable to the verification method. The appropriate value function would be $V(s, r)$ defined as the minimum of the last integral above over the x satisfying

$$\int_0^\infty (x(t))^2\, dt = r, \qquad x(0) = s.$$

A look at the necessary conditions for this problem (we are not sure if they must apply on an infinite horizon, but we are in "guessing mode" anyway) leads us to conjecture a solution of the form $c \exp\{-(t+b)^2\}$, where c and b are chosen to satisfy the two constraints. Solving for these constants and substituting into the integral to be minimized gives a tentative expression for $V(s, r)$:

$$V(s, r) = \frac{1}{2}\{gs^2 + r(1 - g^2 - 2 \ln s)\},$$

where $g = g(s, r)$ equals $h^{-1}(r/s^2)$, h being defined by

$$h(t) = e^{t^2} \int_t^\infty e^{-\tau^2} d\tau.$$

[26] Details appear in [AC1979].

As expected, V is undefined for $s = 0$ and $r \neq 0$. The partial derivatives of V can be computed as

$$V_s = g_s, \qquad V_r = -\frac{g^2}{2} - \ln s.$$

Armed with this, we easily confirm the inequality

$$\frac{u^2}{2} - s^2 \ln s \geq V_r s^2 - V_s u.$$

We now proceed much as when (3.21) above was invoked. We obtain, for any admissible x,

$$\begin{aligned}\int_0^\infty \left\{ \frac{1}{2}(\dot{x}(t))^2 - (x(t))^2 \ln |x(t)| \right\} dt \\ &\geq -\int_0^\infty \frac{d}{dt} V\left(x(t), \int_t^\infty (x(\tau))^2 d\tau\right) dt \\ &\geq V\left(x(0), \frac{\sqrt{\pi}}{2}\right) - \liminf_{t\to\infty} V\left(x(t), \int_t^\infty (x(\tau))^2 d\tau\right).\end{aligned}$$

The proof is completed by showing that the first term is no less than $\sqrt{\pi}/4$ and that the limit is nonpositive. □

3.8. Value functions as a diagnostic tool in optimal control.

The property of the value function displayed in Proposition 3.1 has been exploited for its role in constructing verification functions, that is, as an inductive tool. But inherent in the property is a necessary condition as well. Specifically, suppose that we form a conjecture regarding the optimal arcs for all values (τ, y) of the terminal boundary condition (or locally at least). Then $V(\tau, y)$ can be calculated provisionally on the basis of the conjecture. If the V so calculated satisfies the hypotheses but not the conclusions of Proposition 3.1, then the conjecture is necessarily incorrect. Futhermore, the very way that V fails to have the necessary property can sometimes be a useful indicator of how to modify the conjecture (and hence V) to make it correct. Thus a feedback system is set up between the necessary and sufficient conditions until the function V itself confirms the optimality of the current conjecture.

Let us illustrate this symbiosis with a problem in optimal control that arises in connection with the modeling of renewable resources. A thorough discussion of the interest in the problem is given in [CCM1979]. Here we shall focus on its mathematical structure and solution. We remark that the known analytical solution of this problem was critical to the development of computational schemes for the numerical study of its behavior, as well as its stochastic extensions, which were too complicated for an analytical solution.

The problem has two state variables (x, K) (biomass and capital) and two control variables (u, I) (harvesting effort and investment) linked by the dynamics

$$\dot{x}(t) = F(x(t)) - \beta u(t) x(t), \qquad \dot{K}(t) = I(t) - \gamma K(t),$$

where F is a given function and β, γ given constants. The values of x and K at $t = 0$ are given, and $u(t)$, $I(t)$ are constrained to the intervals $[0, K(t)]$ and $[0, \infty)$, respectively. The admissible state arcs x and K are absolutely continuous, except that K is allowed to have a countable number of jumps (this is viewed as allowing $I = +\infty$).

The quantity to be maximized is the net discounted return corresponding to a control policy (u, I) and associated state (x, K), given by

$$\int_0^\infty e^{-\delta t}\{(px(t) - c)u(t) - \pi I(t)\}\, dt - \sum_{i=0}^{\infty} e^{-\delta t_i}[K(t_i+) - K(t_i-)],$$

where the second term is the cost of the jumps occurring at the (variable) times t_i, and where δ, p, π, and c are given positive parameters (the discount rate, price, investment cost, and effort cost). Note that this problem has an infinite horizon.

The necessary conditions for this problem, which we do not describe, identify two distinguished levels of the state x, labeled x^* and $\tilde{x}$. Let $K^* = F(x^*)/(\beta x^*)$. Economic reasoning leads us to conjecture that when $(x(0), K(0)) = (x, K)$ is close to (x^*, K^*), with $x < x^*$, the optimal strategy is to employ the control values $u = K(t)$, $I = 0$ until the time $t = \tau(x, K)$ when $x(\tau) = x^*$ (point E on FIG. 2.

Then K jumps (instantaneously) to $K = K^*$, and thereafter (x, K) stays at (x^*, K^*) (point F) by employing the appropriate constant control values (i.e., $u = K^*$, $I = \gamma K^*$).

Let us denote by $V(x, K)$ the maximum net return when $x(0) = x$, $K(0) = K$. If the conjecture above is correct, then for (x, K) in the appropriate range we can express V as follows:

$$V(x, K) = \int_0^\tau e^{-\delta t}\{(px(t) - c)K(t)\}\, dt + \int_\tau^\infty e^{-\delta t}\{(px^* - c)K^* - \pi\gamma K^*\}\, dt - e^{-\delta\tau}[K^* - K(\tau)], \tag{3.22}$$

where $\tau = \tau(x, K)$ is determined implicitly via the boundary value problem

$$\dot{x}(t) = F(x(t)) - \beta K(t)x(t), \qquad \dot{K}(t) = -\gamma K(t),$$
$$x(0) = x, \quad K(0) = K, \quad x(\tau) = x^*.$$

From the very definition of the value function V, assuming smoothness and using arguments similar to those used to prove Proposition 3.1, we can deduce that V should satisfy the following Hamilton–Jacobi inequality:

$$\delta V + u[\beta x V_x - p\beta x + c] - F(x)V_x + \gamma K V_K + I(\pi - V_K) \geq 0. \tag{3.23}$$

This implies, in particular, the global inequality $V_K(x, K) \leq \pi$.

If we examine the function V defined by (3.22), we can show that the equation $V_K = \pi$ defines a curve S_2 through (x^*, K^*) positioned as indicated in Fig. 2. Below and to the right of S_2, V satisfies $V_K > \pi$, in violation of the necessary condition $V_K \leq \pi$. This is an indication that the strategy as currently defined is not optimal in that region. So we redefine it in that region only, by defining

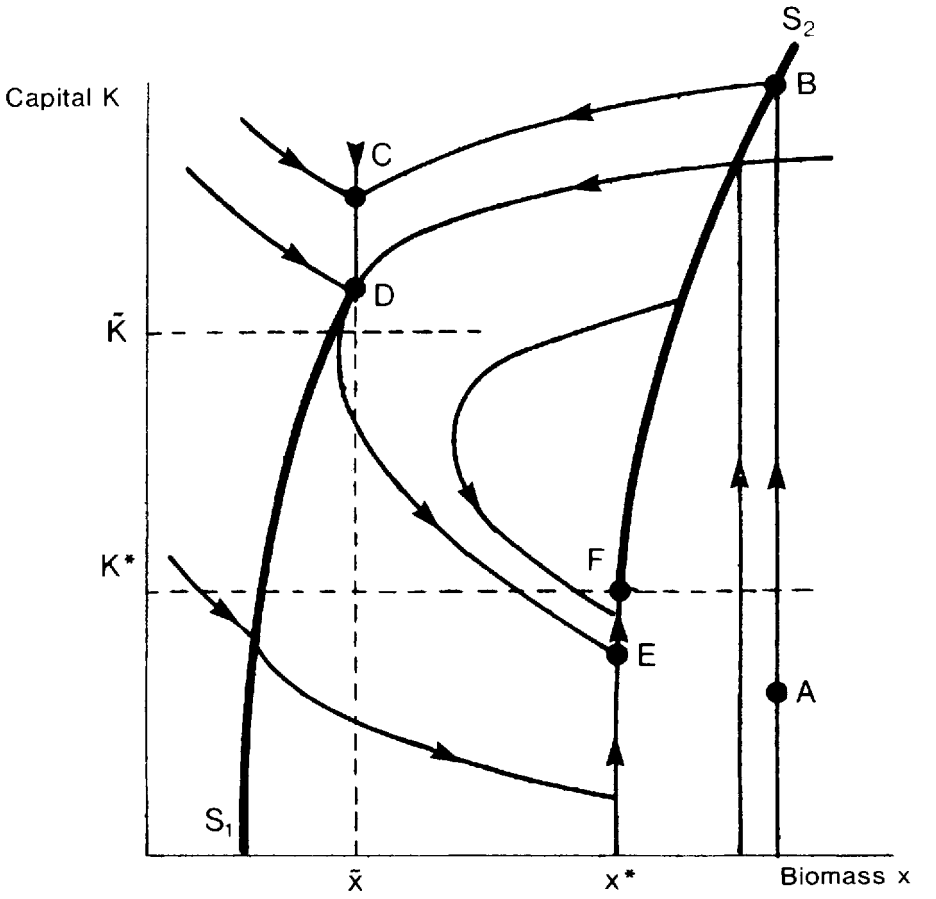

FIG. 2

the policy from an initial point (x, K) below S_2 as follows: jump immediately up to S_2; then follow the policy previously prescribed.

This leads to a new function V, continuous but not smooth across S_2. We can show that where V_K exists, it satisfies $V_K \leq \pi$ as required. Turning anew to (3.23), we find embodied in it a constraint involving V_x. This leads to another curve S_1, on the other side of which the putative optimal strategy (and hence V) must again be redefined.

We shall not describe this in any further detail, since we wish only to illustrate the role played by the Hamilton–Jacobi inequality in deducing the solution and, of course, in confirming it. Once the process above is complete, the final function V confirms the optimality of the strategy producing it. A synthesis of this strategy is given in Fig. 2.

The utility of having proved analytically the nature of the optimal solution is highlighted by the improbability, at first glance, of some aspects of it. Consider, for example, an initial value (x, K) having large x, small K (point A). Initially the solution jumps vertically to S_2, then moves left until $x = \tilde{x}$, $K > \widetilde{K}$ (point C). Then the solution lingers at $x = \tilde{x}$ for some time, until point D is reached, only to dip below $\tilde{x}$ subsequently, then builds up again to $x = x^*$. At this point there is a vertical jump to (x^*, K^*), where the solution stays thereafter. This kind of solution becomes evident to economists only sometime after they are convinced that it must be accepted.

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CHAPTER 4

Optimal Control

4.1. The maximum principle.

The central role played by the Hamiltonian in the classical calculus of variations (a phrase almost synonymous with classical mechanics) is well known. In that setting, the Hamiltonian H is defined by applying the Legendre transform to a smooth Lagrangian L, as follows. Impose conditions so that the equation $p = L_v(t, x, v)$ defines an implicit function $v = v(t, x, p)$; then set

$$H(t, x, p) = \langle p, v \rangle - L(t, x, v),$$

where $v = v(t, x, p)$.

Besides being essentially local in nature and requiring quite a lot of smoothness, this definition suffers from one very serious defect: it does not extend to problems involving side (auxiliary) constraints, such as the isoperimetric condition $\int_0^T G(t, x, \dot{x})dt = 0$, the unilateral velocity constraint $\dot{x} \in U$, the pointwise equality $\psi(t, x, \dot{x}) = 0$, or the differential inclusion $\dot{x} \in E(x)$. Thus the calculus of variations has offered no Hamiltonian theory for problems with auxiliary constraints, despite the growing recognition in the first half of the twentieth century of the importance of these problems in modeling.

In fact, such problems became the very basis of the theory of optimal control, the development of which can be viewed as a paradigm shift from that of the calculus of variations. Interestingly, the word "Hamiltonian" is often used in connection with the theorem that launched the new subject: the maximum principle of Pontryagin. This result bears upon controlling systems of the type

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.,} \qquad t \in [0, T] \tag{4.1}$$

$$u(t) \in U \quad \text{a.e.,} \qquad t \in [0, T]. \tag{4.2}$$

The point is to choose a control (function) u that satisfies (4.2) and generates via the dynamics (4.1) a state (function) x, so as to minimize a cost functional

$$g(x(0), x(T)) + \int_0^T F(t, x(t), u(t))dt. \tag{4.3}$$

We generally impose certain endpoint constraints as well:

$$(x(0), x(T)) \in S. \tag{4.4}$$

Let us define the pseudo-Hamiltonian $\widetilde{H}(t, x, p, u)$ as follows:

$$\widetilde{H}(t, x, p, u) := \langle p, f(t, x, u)\rangle - F(t, x, u). \tag{4.5}$$

Then the maximum principle (which applies to smooth data) states that corresponding to an optimal process (i.e., state and control pair (x, u)), there is an arc p such that[27]

$$-\dot{p}(t) = \widetilde{H}_x(t, x(t), p(t), u(t)) \quad \text{a.e.}, \tag{4.6}$$

$$\dot{x}(t) = \widetilde{H}_p(t, x(t), p(t), u(t)) \quad \text{a.e.}, \tag{4.7}$$

$$\widetilde{H}(t, x(t), p(t), u(t)) = \max_{w \in U} \widetilde{H}(t, x(t), p(t), w) \quad \text{a.e.}, \tag{4.8}$$

$$(p(0), -p(T)) - \nabla g(x(0), x(T)) \text{ is normal to } S \text{ at } (x(0), x(T)). \tag{4.9}$$

There is something decidely reminiscent of a Hamiltonian system here: the visual resemblance of (4.6) and (4.7) to (3.13a) and (3.13b). This undoubtedly explains why many believe that the maximum principle extends the Hamiltonian approach to optimal control problems. Yet there is no Hamiltonian nature to the result at all.

This can be seen, for example, by considering the problem (P) featured in the two preceding chapters. It is clearly a special case of the optimal control problem, obtained by setting

$$\begin{aligned} &f(t, x, u) = u \quad (\text{hence } \dot{x} = u), \\ &U = \mathbb{R}^n, \qquad F(t, x, u) = L(t, x, u), \\ &S = \{(x_0, x_T)\}, \qquad g \equiv 0. \end{aligned}$$

Then (4.7) is a tautology (as it is in all cases), (4.8) is the classical Weierstrass condition (implying $p(t) = L_v(t, x(t), \dot{x}(t))$), and (4.6) is the Euler equation. The function $\widetilde{H}(t, x, p, \dot{x})$ embodies no duality or transform technique, depending as it does on the $\dot{x}$ variable.

The maximum principle was, in fact, the culmination of a long search in the calculus of variations for a comprehensive multiplier rule, which is the correct way to view it : $p(t)$ is a "Lagrange multiplier" for the constraint $\dot{x}(t) - f(t, x(t), u(t)) = 0$. Thus (besides the transversality condition) we write the Euler equation and the Weierstrass condition for the augmented Lagrangian $F + p \cdot \{\dot{x} - f(t, x, u)\}$; these are the conditions of the maximum principle. Of course, the Weierstrass

[27] We state here only the conclusion in *normal* form. In general, g and F are multiplied by a factor λ equal to 0 or 1, with $p \neq 0$ in the former case.

condition is relative to the given set U, which can be closed or even discrete; this is a technical breakthrough of great importance. The change in emphasis from the minimizing state x, viewed as a natural and God-given object, to the control u and the attendant element of choice may have been just as important. This is a philosophical but nonetheless important point. It makes optimal control a design tool, whereas the calculus of variations was a way to study nature. And it naturally shifts attention to the behavior of systems and to new issues such as controllability.

To develop a true Hamiltonian theory, we first need a workable definition. The key is to use the Fenchel transform of convex analysis, which generalizes (more or less) the Legendre transform. For the basic problem (P) we get

$$H(t,x,p) := \max_v \{\langle p, v\rangle - L(t,x,v)\}. \tag{4.10}$$

This expression is more natural than we might think. Recall, for example, how inevitably we were led to it in §3.2 while discussing verification functions, and in §2.6 in describing growth properties of the Lagrangian in connection with existence theory. Note also that the equation $p = L_v(t,x,v)$ that defines the Legendre transform is the necessary condition associated with the maximization problem (4.10).

The Hamiltonian for problems with auxiliary constraints is calculated by expressing the problem in the form of (P). We do this through a suitably chosen *extended-valued* Lagrangian L that incorporates the constraints implicitly, and then we define H via (4.10). (The use of extended-valued functions to incorporate constraints was discussed in §1.3.) Some examples will make this clear.

Consider first the problem of minimizing the variational functional

$$\int_0^T F(t,x(t),\dot{x}(t))dt$$

over the arcs x satisfying $\dot{x}(t) \in U$ almost everywhere, and also boundary constraints of the usual type. We define

$$L(t,x,v) = \begin{cases} F(t,x,v) & \text{if} \quad v \in U, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the stated problem is equivalent to that of minimizing $\int_0^T L(t,x,\dot{x})dt$ (subject to the given boundary conditions). We remark that integrals of extended-valued functions repose on a rigorous theory, and that the equivalence is in a strict sense, not merely heuristic. The problem (P) in which L is allowed to be extended-valued is called the *generalized problem of Bolza.* We shall refer to it as (P_B) to maintain the distinction.[28]

It can be shown that the standard optimal control problem (4.1)-(4.4), which we shall refer to as (P_C), is equivalent under very mild conditions to a generalized problem of Bolza in which L is given by

$$L(t,x,v) = \inf\{F(t,x,u) : \ u \text{ in } U \text{ satisfying } f(t,x,u) = v\} \tag{4.11}$$

[28] In general (P_B) also incorporates an extended-valued term $\ell(x(0),x(T))$ to reflect all boundary costs and constraints; see §4.6.

(the infimum is $+\infty$ by the usual convention when no such u exist). We leave as an exercise for the reader the fact that when L is given by (4.11), then the Hamiltonian H defined by (4.10) becomes

$$H(t,x,p) = \max_{u\in U}\{\langle p, f(t,x,u)\rangle - F(t,x,u)\}. \tag{4.12}$$

This expression is not unknown in the classical theory of optimal control, in view of the condition (4.8) in which it implicitly appears. The H of (4.12) has sometimes been referred to as the "maximized Hamiltonian"; we know it as simply the Hamiltonian.

Let us calculate one further Hamiltonian, the one for the following differential inclusion problem, denoted (P_D):

$$\begin{aligned} &\text{minimize } g(x(0),x(T)) \\ &\text{subject to } \dot{x}(t)\in E(t,x(t)) \text{ a.e., } (x(0),x(T))\in S. \end{aligned} \tag{4.13}$$

The Lagrangian corresponding to this problem is

$$L(t,x,v) = \begin{cases} 0 & \text{if } v\in E(t,x), \\ +\infty & \text{otherwise,} \end{cases}$$

and the Hamiltonian easily is seen to be

$$H(t,x,p) = \max_{v\in E(t,x)} \langle p, v\rangle. \tag{4.14}$$

The differential inclusion (P_D), necessary conditions for which were first developed in [C1973], actually subsumes the optimal control problem (P_C) and certain more general problems as well (such as when the control set $U(x)$ depends on x (feedback)). (P_D) is at the opposite extreme from (P_B) — which is all cost function in appearance — by its being almost entirely dynamic constraint. For this reason, (P_D) is especially well suited to issues such as controllability. On the other hand, (P_B) is a better paradigm for existence theory, for example, since it lends itself well to expressing growth properties. We shall elaborate on these remarks later. For now we shall begin our study of necessary conditions in Hamiltonian form in the context of differential inclusions.

4.2. Hamiltonian necessary conditions.

Let us consider the differential inclusion problem (P_D); for ease of presentation we make some simplifying assumptions. We take the set S of (4.13) to be of the form $C_0\times\mathbb{R}^n$, where C_0 is a given closed subset of $\mathbb{R}^n$. Thus $x(0)$ is constrained to lie in C_0 while the endpoint $x(T)$ is free. It turns out that in optimal control problems an entire level of complexity results from having both endpoints constrained. We shall be discussing ways to treat these difficulties presently. We also suppose that E is autonomous (no t dependence), that $E(x)$ is compact-valued and Lipschitz in x (as a multifunction in the Haussdorf metric), and that the cost g is locally Lipschitz. (The Lipschitz condition on E requires that for some k, for all x and y, the set $E(x)$ is contained in $E(y)+kB$.)

We introduce the function $\rho(x,v) = d_{E(x)}(v)$. We observe that for any arc x, the quantity $\int_0^T \rho(x(t),\dot{x}(t))dt$ may be thought of as a measure of the extent to

which x fails to be a trajectory for E (i.e., a solution of the differential inclusion in (4.13)). In fact, the quantity in question vanishes if and only if x is a trajectory. With this in mind, the following result naturally compares to the penalization technique of Proposition 1.1 of Chapter 1, although in this case, the proof (which we omit; see [C1973], [C1975c]) is much less immediate.

PROPOSITION 4.1. *Let the arc x solve the differential inclusion problem (P_D) under the hypotheses stated above. Then for some constant $K > 0$, x is a (strong) local minimum for*

$$g(y(0), y(T)) + K \int_0^T \rho(y(t), \dot{y}(t))dt$$

over the arcs y that satisfy $y(0) \in C_0$.

The consequence of this result is that a reduction is achieved under which x is now a solution of a more or less standard variational problem, one in which the Lagrangian ρ is locally Lipschitz but not differentiable. Theorem 2.2 applies here to yield the existence of an arc p satisfying, together with certain transversality conditions, the Euler inclusion

$$(\dot{p}(t), p(t)) \in \partial\rho(x(t), \dot{x}(t)) \quad \text{a.e.}$$

If the multifunction E satisfied certain additional hypotheses of smoothness and fullness, this would immediately yield the "Hamiltonian inclusion" below. In general, however, we must resort to an approximation and convergence procedure. Here is the outcome (H is of course defined by (4.14)):

THEOREM 4.1. *Under the hypotheses of Proposition 4.1, there is an arc p satisfying the Hamiltonian inclusion*

$$(-\dot{p}(t), \dot{x}(t)) \in \partial H(x(t), p(t)) \quad a.e., \tag{4.15}$$

$$(p(0), -p(T)) \in \partial g(x(0), x(T)) + N_{C_0}(x(0)) \times \mathbb{R}^n, \tag{4.16}$$

$$H(x(t), p(t)) = \text{constant on } [0, T].^{29} \tag{4.17}$$

Note that when H is smooth, (4.15) reduces to a Hamiltonian system in the classical sense, as in (3.13) of §3.2. Of course, there is no classical differential inclusion problem to refer to, but we can apply the theorem to the special case of the optimal control problem (P_C), under analogous Lipschitz behavior and free endpoint hypotheses. (The fact that (P_C) is subsumed by (P_D) is a well-known measurable selection assertion called Filippov's lemma.) The result is an extension of the maximum principle to nonsmooth data.[30]

[29]It is a recurrent fact that transversality conditions can be given in terms of presubdifferentials (see the remarks following the proofs of Theorems 1.1 and 2.2). Thus (4.16) can be stated with $\hat{\partial}g$ and $\hat{N}_{C_0}$. Analogous facts also hold for future results too, but we shall not make them explicit.

[30]The derivation of the corollary from the theorem is given in [C1979b]; the general maximum principle for nonsmooth data and arbitrary endpoint constraints first appeared in [C1976b].

COROLLARY 4.1. *Let* (x,u) *solve* (P_C). *Then there is an arc* p *satisfying* (4.16) *together with*

$$-\dot{p}(t) \in \partial_x \widetilde{H}(x(t),p(t),u(t)) \quad a.e., \tag{4.18}$$

$$\dot{x}(t) = \widetilde{H}_p(x(t),p(t),u(t)) \quad a.e., \tag{4.19}$$

$$\widetilde{H}(x(t),p(t),u(t)) = H(x(t),p(t)) = \textit{constant} \quad a.e. \tag{4.20}$$

We remark that (4.18) implies

$$-\dot{p}(t) \in p(t)\partial_x f(x(t),u(t)) - \partial_x F(x(t),u(t)), \tag{4.21}$$

where $\partial_x f$ refers to the generalized Jacobian of f in x (see §1.5). Thus we see that for (P_C), the Hamiltonian necessary conditions given by the theorem imply the non-Hamiltonian conditions of the maximum principle, in the sense that the latter are easily deducible from the former, but not in the strict sense that (x,p,u) satisfies (4.18) and (4.19) whenever (x,p) satisfies (4.15). Still, the single relation (4.15) essentially captures the state equation (4.19) (or equivalently (4.1)), the "adjoint equation" (as the smooth version of (4.21) is often called), and the maximum condition (4.20) (or (4.8)), while the passage in the other direction does not seem possible. The Hamiltonian form speaks only of optimal state arcs x, and not the control(s) u that correspond to them. This "deparametrized" form of the necessary conditions benefits by being intrinsic; it depends only upon the multifunction (which, when $F \equiv 0$, is $E(x) = f(x,U)$), and not the particular f, U of the representation. We shall see later how this gives more precise information on the value function than the maximum principle form, which is representation-specific. Of course, we can imagine that being representation-specific could be advantageous in some circumstances, such as when the nonfulfillment of necessary conditions is invoked to rule out optimality. In fact, examples exist to show that in specific cases either of these different sets of necessary conditions can be more precise than the other.

4.3. Attainable sets and controllability.

We now wish to address optimal control problems with full endpoint constraints. To do so, it turns out to be useful, if not inevitable, to consider associated issues of controllability. An object of central importance is the *attainable set* $\mathcal{A}(T,x_0)$ (at T, from x_0) defined as follows:

$$\mathcal{A}(T,x_0) := \{x(T) : x \text{ is a trajectory for } E \text{ on } [0,T] \text{ with } x(0) = x_0\}. \tag{4.22}$$

It has long been known[31] that "boundary arcs" x satisfy conditions very much akin to the necessary conditions, and that conical approximations to $\mathcal{A}(T,x_0)$ play a role in developing these conditions.

We now describe an approach pioneered in [C1976b] for obtaining necessary conditions for trajectories x such that $x(T)$ lies on the boundary of $\mathcal{A}(T,x_0)$. This approach uses a perturbation argument involving Ekeland's theorem together with nonsmooth analysis and a convergence procedure. Numerous authors have since adapted it to a variety of other situations. The crux of the

[31] See, for example, the original proof of the maximum principle [PBGM1962].

approach is particularly apparent in the present simple context. We now add to the hypotheses on E of the previous section that it is convex-valued; this assures that $\mathcal{A}(T, x_0)$ is closed.

Now suppose that x is a trajectory for E on $[0, T]$ with $x(0) = x_0$ such that $x(T)$ lies on the boundary of $\mathcal{A}(T, x_0)$. For any $\epsilon > 0$, there is a point ζ outside $\mathcal{A}(T, x_0)$ but within distance ϵ of $x(T)$. For such a point ζ, consider now the problem

$$\text{minimize } |\zeta - y(T)| \text{ over the trajectories } y \text{ for } E \text{ satisfying } y(0) = x_0.$$

Observe that the choice $y = x$ must "almost solve" this problem, since its value is necessarily nonnegative, and since $|\zeta - x(T)| < \epsilon$. However, we have no reason to believe that x actually solves the problem, so we are in the position of wanting to conclude something about an almost-optimal arc.

There is a result, now rather well known, that bears upon this situation. Ekeland's theorem states[32] (roughly speaking) that there is a slightly perturbed problem that does admit a solution at an arc that is close to x. In the present situation, a precise statement of this principle is that some trajectory x_ζ exists that solves the problem

$$\text{minimize } |\zeta - y(T)| + \sqrt{\epsilon}\int_0^T |y(t) - x_\zeta(t)|dt \text{ over trajectories } y \text{ for } E \text{ satisfying } y(0) = x_0$$

and that satisfies

$$\int_0^T |x_\zeta(t) - x(t)|dt < \sqrt{\epsilon}.$$

Note that the problem that x_ζ solves is a free endpoint one, and that the data are nonsmooth but Lipschitz. The necessary conditions of Theorem 4.1 can be applied following some simple reformulating of the problem to give it an appropriate guise (for instance to account for its not being autonomous). We deduce that x_ζ, together with some arc p_ζ, satisfies a Hamiltonian inclusion that involves the perturbed Hamiltonian $H_\zeta(t, x, p) = H(x, p) - \sqrt{\epsilon}|x - x_\zeta(t)|$. Because $|\zeta - x_\zeta(T)| \neq 0$ (why?), the transversality condition gives $|p_\zeta(T)| = 1$, which implies nontriviality of the adjoint arc p.

If the step above is performed for a sequence of ϵ_i decreasing to 0, then the associated (ζ_i, x_{ζ_i}) converge to $(x(T), x)$ and, at least for a subsequence the p_{ζ_i}, converge to a nontrivial arc p. Because the subdifferential is upper semicontinuous, ∂H_{ζ_i} converges to ∂H appropriately, and it follows that p is an adjoint arc for x. We summarize in the following theorem.[33]

[32] See [E1974], [E1979], as well as [BP1987], for a related result.

[33] We tend to believe that this theorem remains true without the hypothesis of convexity on the multifunction, but it has resisted the efforts of several authors. The analogous result in maximum principle form has been proven in [C1976b]. See [F1987] for related developments.

THEOREM 4.2. *If x is a trajectory for E satisfying $x(0) = x_0$ such that $x(T)$ lies on the boundary of $\mathcal{A}(T, x_0)$, then there exists a nonzero arc p satisfying $(-\dot{p}, \dot{x}) \in \partial H(x, p)$ almost everywhere on $[0, T]$.*

When it is suitably extended to apply to initial sets C_0 more general than $\{x_0\}$, and to images of $\mathcal{A}(T, C_0)$, the theorem becomes a tool to obtain necessary conditions, since optimality always implies that some augmented state related to x is a boundary trajectory of the multifunction E augmented by a cost function component. When full endpoint constraints are present, the necessary conditions may hold only in a degenerate form, termed *abnormal*, in which the cost function does not figure. This occurs, roughly speaking, when (due to lack of controllability of the underlying system) fulfilling the constraints is so demanding that the cost function becomes irrelevant. Here is such a result for the inclusion problem (P_D) given by (4.13); it is an easy consequence of Theorem 4.2.

THEOREM 4.3. *If x solves (P_D), then there exist a scalar λ equal to 0 or 1, and an arc p satisfying the Hamiltonian inclusion with x, such that*

$$(p(0), -p(t)) \in \lambda \partial g(x(0), x(T)) + N_S(x(0), x(T)). \tag{4.23}$$

If λ is 0, then p is nonvanishing.

(*Note.* The last assertion is essential, because $\lambda = 0$ and $p \equiv 0$ trivially satisfy the necessary conditions.)

A trajectory x for E admissible for (P_D) is called *normal* if there is no nontrivial adjoint arc p for x satisfying (4.23) for $\lambda = 0$. The problem (P_D) is called normal if all of its solutions are normal. Free endpoint problems are automatically normal. We shall see presently that the normality of a trajectory relative to a set S implies a controllability and stability property in terms of translates of S. The key to this result is the study of certain value functions defined by perturbations of (P_D). We turn now to this subject.

4.4. Value functions and Hamiltonian multipliers.

We have pointed out in Chapter 1 the general effect of a proximal normal to the epigraph of a value function. When the defining inequality is expressed, a version of the underlying problem emerges in which the constraint that is being perturbed is transformed into an additional term in the cost. In view of this, and since the removal of an endpoint constraint greatly decreases the level of complexity of an optimal control problem, we are naturally led to explore the following value function:

$$V(\alpha) := \min \{g(x(0), x(T)) : \ x \text{ is a trajectory for } E \text{ satisfying } (x(0), x(T)) \in S + \alpha\}.$$

Let us think of the perturbed constraint as an equality, $(x(0), x(T)) = s + \alpha$, for some s in S. Then (review the first step in the proof of Theorem 1.1), if β is a proximal subgradient of V, the defining proximal inequality leads to an augmented cost of the form

$$g(x(0), x(T)) - \beta \cdot (x(0), x(T)) + \beta \cdot s.$$

We can now invoke the free endpoint case (Theorem 4.1) to deduce the existence of an adjoint arc p. The transversality condition gives

$$(p(0), -p(T)) \in \partial g(x(0), x(T)) - \beta,$$

while the freedom to choose s in S leads to $\beta \in -N_S$. We have waved our hands to waive a lot of detail here, in particular, a convergence procedure that leads to solutions of the $\alpha = 0$ problem in the limit. The main point is twofold: proximal analysis allows us to reach the fully constrained endpoint case from the free endpoint case; in so doing a formula for $\partial V(0)$ results as well. Let us note this formula precisely.

For any trajectory x, we denote by $M(x)$ the set of *Hamiltonian multipliers* for x; that is, all arcs p satisfying the following normal necessary conditions

$$(-\dot{p}, \dot{x}) \in \partial H(x, p) \quad \text{a.e. on } [0, T],$$

$$(p(0), -p(T)) \in \partial g(x(0), x(T)) + N_S(x(0), x(T)).$$

For any element p of $M(x)$, $\theta(p)$ signifies the (nonempty) set

$$-N_S(x(0), x(T)) \cap \{\partial g(x(0), x(T)) - (p(0), -p(T))\}.$$

Finally, Σ signifies the set of arcs x solving the problem (P_D), given, as usual, by (4.13). The notation $\theta M(\Sigma)$ denotes

$$\{\beta : \beta \in \theta(p) \text{ for some } p \in M(x), \text{ for some } x \in \Sigma\}.$$

THEOREM 4.4. *If (P_D) is normal, then V is finite and Lipschitz near 0, and we have*

$$\partial V(0) = \text{co } \{\partial V(0) \cap \theta M(\Sigma)\}.$$

Note the resemblance of this formula to that of Theorem 1.1, which is no accident. Just as for Theorem 1.1, a number of consequences flow from Theorem 4.4 that concern directional derivatives of V, etc. Let us note just one of these. If Σ consists of a single arc x, if $M(x)$ consists of a single arc p, and if g is smooth, then $\nabla V(0)$ exists and equals $\nabla g(x(0), x(T)) - (p(0), -p(T))$. This follows directly from the formula since, under those hypotheses, $\theta M(\Sigma)$ is a singleton, and thus so is $\partial V(0)$ (which is nonempty since V is Lipschitz near 0).

Theorem 4.4 is not just a formula for $\partial V(0)$; it actually subsumes the necessary conditions of Theorem 4.3. For if x solves (P_D), we can consider a modified problem that has an additional cost term $\int_0^T |y(t) - x(t)|^2 dt$ (which requires a bit of reformulation). x is now the unique solution and the set $M(x)$ remains unchanged. If this problem is abnormal, then the abnormal conclusion of Theorem 4.3 results. Otherwise, we apply Theorem 4.4 to this new problem. Then the fact that the new $\partial V(0)$ is nonempty implies that $M(x)$ is nonempty, which is the normal form of the conclusion of Theorem 4.3.

Another consequence of Theorem 4.4 is a sufficient condition for stability and controllability, whose proof once again involves careful bookkeeping for a modified problem to which Theorem 4.4 is applied [C1983, Thm. 3.5.3].

THEOREM 4.5. *If x is a normal trajectory for E, then for a certain constant κ, for every α in $\mathbb{R}^{2n}$ near 0, there is a trajectory y for E satisfying*

$$(y(0), y(T)) \in S + \alpha$$

and

$$\int_0^T |x(t) - y(t)|dt \leq \kappa|\alpha|.$$

The parameters that undergo perturbation in this section have related to the endpoint constraints; they represent a natural and useful choice. When the problem arises from a specific context, it is almost certain that additional parameters will suggest themselves, and hence that other value functions will be of potential interest. We can imagine, therefore, a parametrized problem along the following lines:

$$\text{minimize } g(x(0), x(T), \alpha) \text{ over the arcs } x \text{ on } [0,T]$$
$$\text{satisfying } \dot{x}(t) \in E(x(t), \alpha) \quad \text{a.e. and} \quad (x(0), x(T), \alpha) \in S.$$

A general formula for the subdifferential of the resulting value function $V(\alpha)$ has been obtained by Clarke and Loewen.[34] We can extend the definition of Hamiltonian multipliers to this context and use them as above to generate a formula for $\partial V(0)$ and attendant corollaries. The main restriction in these results is that the parameter α must be finite-dimensional. The application of proximal normal analysis in an infinite-dimensional context is more recent and is the next topic.

4.5. An infinite-dimensional perturbation.

The application of proximal analysis to infinite-dimensional perturbations requires the extension of the proximal normal formula, the theorem that the normal cone N_C is generated by limiting proximal normals. The extension due to Borwein and Strojwas [BS1986] (see §1.6) recently has made possible a number of results, the first of which we now discuss. The context will be the standard autonomous optimal control problem (P_C) given by (4.1)–(4.4), in the case when $S = \{x_0\} \times \mathbb{R}^n$ (i.e., $x(0)$ is prescribed and $x(T)$ is free), so that g can be taken to depend on $x(T)$ only. As usual, locally Lipschitz behavior in x is postulated.

We consider the perturbed problem $P_C(\alpha)$, where α is a function in $L^2([0,T]; \mathbb{R}^n)$, in which the dynamics (4.1) are replaced by

$$\dot{x}(t) = f(x(t), u(t)) + \alpha(t) \quad \text{a.e.} \quad t \in [0,T].$$

The value of this perturbed problem is $V(\alpha)$. For any state trajectory x (corresponding to some control u) for the original problem ($\alpha = 0$) we denote by $M(x)$ the set of Hamiltonian multipliers p for x: that is, the set of arcs p satisfying the Hamiltonian inclusion with x (where H is given by (4.12)) and the transversality condition $-p(t) \in \partial g(x(T))$ (such necessary conditions are a special case of Theorem 4.1). As usual, Σ signifies the set of state solutions to the

[34] See [CL1986], where the horizon T is allowed to vary as well, a type of consideration we postpone until §4.8.

original unperturbed problem. Since V is defined on the Hilbert space L^2, we may consider $\partial V(0)$ to be a subset of L^2 as well.

THEOREM 4.6. *V is finite and Lipschitz in a (strong) neighborhood of 0 in L^2; $\partial V(0)$ consists of arcs, and we have*

$$\partial V(0) = \operatorname{cl}\operatorname{co}\{-M(\Sigma) \cap \partial V(0)\}.$$

Given the frequency with which we have now seen such formulas, it might tax the reader's patience to make explicit more than one of its consequences. We select a sufficient condition for differentiability.[35] If there is a unique solution x and a unique element p in $M(x)$, then the strict derivative $DV(0)$ exists and equals $-p$. We remark that the analogous result for extremals in the sense of the maximum principle fails. There exist smooth problems admitting a unique optimal process (x, u) and a unique associated adjoint variable p in the sense of the maximum principle (4.6)–(4.9) such that V is not differentiable at 0.

Example. Let us illustrate the use of Theorem 4.6 in a simple ballistics problem. Let $[x_1(t), x_2(t)]$ be the coordinates in the $x - y$ plane of a projectile launched from rest from the origin at $t = 0$. If a constant thrust τ is employed with a variable steering angle $u(t)$, we obtain the equations of motion

$$\begin{aligned}\dot{x}_1 &= x_3,\\ \dot{x}_2 &= x_4,\\ \dot{x}_3 &= \tau \cos u,\\ \dot{x}_4 &= \tau \sin u - \overline{g},\end{aligned}$$

where x_3, x_4 represent the velocity components and $\overline{g}$ is the acceleration of gravity ($\tau > \overline{g}$). We suppose that fuel runs out at $t = T$, so that thereafter the projectile is subject only to gravity until it lands (i.e., until $x_2 = 0$ occurs). The goal is to maximize total horizontal range, i.e., $x_1(T)$ plus the horizontal distance traveled after fuel runs out. Equivalently we seek to minimize the negative of the range, a quantity g that can be calculated in terms of $x(T)$:

$$g = -x_1(T) - x_3(T)\{x_4(T) + [x_4(T)^2 + 2\overline{g}\, x_2(T)]^{1/2}\}/\overline{g}.$$

We recognize this as a special case of the problem treated in this section, with $x_0 = (0, 0, 0, 0)$, $F \equiv 0$, $U = [0, \pi/2]$ (say), and g as above. The Hamiltonian inclusion is readily analyzed, and the necessary conditions identify the unique solution to the problem [C1986]. It turns out that the optimal steering policy $u(t)$ is a constant $\overline{u}$ determined by a certain equation.

In calculating the solution to the problem (which is a standard exercise in optimal control), we must calculate at the same time the (unique) adjoint variable $p(t)$, which turns out to have the following form (if we let m signify $\sqrt{x_4(T)^2 + 2\overline{g}\, x_2(T)}$):

$$\begin{aligned}p(t) = \big[1, \tan \overline{u}, &- t + T + \overline{g}^{-1}\{x_4(T) + m\},\\ &- (\tan \overline{u})(t - T) + \overline{g}^{-1} x_3(T)\{1 + m^{-1} x_4(T)\}\big].\end{aligned}$$

[35] This is *strictly* speaking; see [C1983, §2.2].

Generally this function is treated as just an inconsequential by-product of the procedure. But armed with Theorem 4.6, we are able to exploit it for sensitivity analysis.

For example, suppose now there is a wind effect, so that only the velocity of the projectile is altered:

$$\dot{x}_1(t) = x_3(t) + \alpha_1(t), \qquad \dot{x}_2(t) = x_4(t) + \alpha_2(t).$$

(We have $\alpha_3 = \alpha_4 \equiv 0$.) What effect will this have on the maximum range?

The perturbed problem is nonautonomous, and the simple closed-form solution we had for the unperturbed problem is no longer available. But by the theorem, V has derivative $-p(\cdot)$ at 0, so for small α (in the L^2 norm), we have

$$\begin{aligned} V(\alpha) &\cong V(0) - \langle \alpha, p \rangle \\ &= V(0) - \int_0^T \alpha_1(t)dt - \tan \overline{u} \int_0^T \alpha_2(t)dt. \end{aligned}$$

We can use this to verify the correctness of some numerical procedure to calculate $V(\alpha)$ or for certain qualitative conclusions. For example, the effect of a small wind will increase the range (i.e., decrease V) if the mean wind $(\overline{\alpha}_1, \overline{\alpha}_2)$ points in the direction $(1, \tan \overline{u})$, i.e., if the inner product of these vectors is positive.

As another example, suppose we can supplement the thrust by a time-varying $[\alpha_3(t), \alpha_4(t)]$ subject to $\int_0^T (\alpha_3^2 + \alpha_4^2)dt \leq \delta$, where δ is small. What choice should be made?

We have (setting this time $\alpha_1 = \alpha_2 \equiv 0$)

$$V(\alpha) - V(0) \simeq -\int_0^T (\alpha_3, \alpha_4) \cdot (p_3, p_4)dt,$$

where the functions p_3 and p_4 are given above. Upon minimizing this subject to the given constraint on (α_3, α_4), we find the best increment of thrust to be

$$(\alpha_3(t), \alpha_4(t)) = k(p_3(t), p_4(t))$$

for a suitably determined positive constant k. □

4.6. Hamiltonian multipliers in the calculus of variations.

A problem that subsumes all the ones we have looked at— the basic variational problem (P), the differential inclusion problem (P_D), and the optimal control problem (P_C)— is the generalized problem of Bolza:

$$\text{minimize } \ell(x(0), x(T)) + \int_0^T L(t, x(t), \dot{x}(t))dt$$

over the arcs x that map $[0, T]$ to $\mathbb{R}^n$. As we have said earlier, an essential element here is the ability to treat in a sufficiently general setting the case in which ℓ and L are extended-valued. We have discussed in §4.1 how to define L to reflect various kinds of auxiliary constraints, and in §1.3 how to define ℓ to account for various types of boundary constraints and cost functions. A certain price in technical complication must be paid in order to treat so general a problem. However,

existence theory, sufficient conditions, necessary conditions, and regularity can all be successfully addressed via Hamiltonian analysis. We refer to [C1987c] for an account of this theory. Here we limit ourselves to completing the results for the basic problem (P), which occupied center stage in the two preceding chapters.

The Hamiltonian necessary conditions for (P) (under our customary hypotheses, (T1)′–(T3)′ of Chapter 2) can, in fact, be deduced from conditions given earlier for the differential inclusion problem; given that the endpoints are prescribed, they amount in the autonomous case to the existence of an arc p satisfying together with the solution x to (P) the Hamiltonian inclusion $(-\dot{p}, \dot{x}) \in \partial H(x,p)$ almost everywhere and the condition $H(x(t),p(t)) = \text{constant} = h$. We remark that this constancy condition is not a consequence of the inclusion, in contrast to the classical Hamiltonian context.

Chapter 2 had provided its own necessary conditions for optimality, the Euler inclusion $(\dot{p}, p) \in \partial L(x, \dot{x})$ almost everywhere. A question arises: is the arc p in this relation the same as the one in the Hamiltonian inclusion? It turns out that the arcs need not necessarily be the same, so another question arises: can we assert that (for a solution x) a single arc p exists satisfying *both* the Euler and Hamiltonian inclusions? An affirmative answer to this question has been announced by Loewen and Rockafellar [LR1989] (see also [C1989]).

As seen in Chapter 3, a particularly interesting value function related to (P) is

$$V(\tau, y) = \min \int_0^\tau L(x, \dot{x})dt, \quad x(0) = x_0, \quad x(\tau) = y.$$

In the next result, which bears upon V, $M(x)$ refers to the set of multipliers in the following sense: couples $[p,h]$, where p is, together with x, a solution of the Hamiltonian inclusion, and where along (x,p) the Hamiltonian has constant value h. Also, $\theta([p,h])$ signifies $[-h, p(T)]$, and as before, Σ signifies the set of solutions to (P).

THEOREM 4.7. *$V(\tau,y)$ is locally Lipschitz for $\tau > 0$, and we have*

$$\partial V(T, x_T) = \text{ co}\{\partial V(T, x_T) \cap \theta M(\Sigma)\}.$$

The formula can be shown to provide another proof that V satisfies the (generalized) Hamilton–Jacobi equation and also shows that, in a now familiar pattern, multipliers associated to solutions generate pointwise the subdifferential of the value function. But a more global question concerns whether the *same* p can be related to $\partial V(t, x(t))$ all along the arc x. An affirmative answer was given by Clarke and Vinter [CV1987], as shown in the following theorem.

THEOREM 4.8. *If x solves (P), then there is a Hamiltonian multiplier p for x with $H(x(t),p(t)) = h$ such that*

$$(-h, p(t)) \in \partial V(t, x(t)), \qquad 0 \le t \le T.$$

4.7. Controllability of Hamiltonian flows on convex energy surfaces.

Let the set Ω in $\mathbb{R}^{2n}$ be the boundary of a compact convex set containing 0 in its interior; we shall refer to Ω as an *energy surface*. This terminology is inspired

by the usual provenance of Ω as a level set $H(x,p) = h$ of a Hamiltonian. Indeed, the set Ω is endowed with an intrinsic Hamiltonian flow which can be given a specific representation via any function representing Ω this way nondegenerately.

A convenient choice is the *gauge* of Ω, i.e., the function g from $\mathbb{R}^{2n}$ to $[0,\infty)$ defined by $g(z) = \lambda \iff z \in \lambda\Omega$. It is clear that $\Omega = \{z : \ g(z) = 1\}$. If g happens to be differentiable (that is, if Ω is smooth), then we define a system of Hamiltonian equations via

$$J\dot{z}(t) = \nabla g(z(t)), \tag{4.24}$$

where the $2n \times 2n$ matrix J was defined in §2.8. Note that the solutions to (4.24) beginning on Ω stay on Ω, since g equals a constant along them. Any other representation of Ω would lead to the same solution curves, just differently parametrized. Any curve on Ω that admits a parametrization $z(t)$ satisfying (4.24) is called a *geodesic*. Our proof below applies when Ω is not smooth, without change. It suffices to consider $\partial g(z(t))$ in (4.24).

We are interested in controllability properties of geodesics. In general, of course, we certainly cannot assert that any two points are joined by a geodesic, so the controllability is necessarily of a different kind, namely, with respect to symplectic images. A $2n \times 2n$ matrix M is called *symplectic*, provided it satisfies $M^*JM = J$. The identity matrix is symplectic and provides an interesting and famous special case of the following result. That case has now been proven in a variety of ways, but no proof of the general result is known other than the one given here, which utilizes value functions.

THEOREM 4.9 [C1982a]. *If M is symplectic, then for some ω in Ω there is a nontrivial geodesic that joins ω to $M\omega$.*

Proof. When $H = g$, the dual Hamiltonian $H^*(y,q)$ of §2.8 reduces to the indicator of a certain convex compact set S^* (the polar or Minkowski dual of $S := \operatorname{co}\Omega$) with the property that for every s,

$$\max\{s \cdot \sigma : \ \sigma \in S^*\} = g(s).$$

We consider a family of parametrized problems P_α, where P_α is the problem that consists of minimizing the functional ϕ defined by

$$\phi(z) = -\frac{1}{2}\int_0^1 \langle J\dot{z}, z\rangle dt$$

over the $2n$-dimensional arcs z on $[0,1]$ satisfying $J\dot{z}(t) \in S^*$ almost everywhere and the boundary condition $z(1) = Mz(0) + \alpha$. This is a problem that, when reformulated as a generalized problem of Bolza, has a Lagrangian equal to $-(1/2)\langle J\dot{z}, z\rangle$ plus the indicator of S^*.

We let $V(\alpha)$ denote the minimum in the problem P_α. It can be shown that V is finite and Lipschitz near 0. Now it follows from the definition of V that for any arc z we have

$$\phi(z) \geq V(z(1) - Mz(0))$$

and that equality holds for any arc y solving P_0. To paraphrase, it follows that y minimizes

$$\phi(z) - V(z(1) - Mz(0))$$

over all arcs z satisfying $J\dot{z} \in S^*$ almost everywhere. This is a differential inclusion problem to which the Hamiltonian necessary conditions of Theorem 4.3 apply.

To write the necessary conditions, we need to calculate the Hamiltonian $\overline{H}$ of the problem. We find without difficulty that $\overline{H}(z,p)$ is given by $g(Jp + z/2)$. The Hamiltonian inclusion then gives

$$J\dot{y} = -2\dot{p} \in \partial g(Jp + y/2).$$

It follows that Jp is a translate of $y/2$, and that the function $w = Jp + y/2$ satisfies (4.24).

A solution to (4.24) such as w is automatically such that $g(w(t))$ is constant: $g(w(t)) = h$. Because $V(0)$ is strictly negative, as is easy to see, it follows that y and hence w is nonconstant, whence $h > 0$.

We now turn to the boundary conditions. The transversality condition in the present situation asserts that for some r, we have $p(1) = r, p(0) = M^*r$. Armed with this, we calculate

$$\begin{aligned} Mw(0) &= MJp(0) + My(0)/2 \\ &= MJM^*p(1) + y(1)/2 \\ &= Jp(1) + y(1)/2 = w(1), \end{aligned}$$

as required. The only remaining detail is to account for the fact that w evolves on the surface $g = h$ rather than $g = 1$ $(= \Omega)$. But because g is positively homogeneous, a simple transformation brings w to Ω while preserving its required properties. □

4.8. Varying intervals.

We have been considering various problems over a time interval $[0,T]$, which has been given and is unvarying. Many interesting problems involve the underlying time interval itself as a choice variable. In this final section, we shall examine briefly the applications of the methods of this chapter to some problems of this type, focusing without proof on certain results of Clarke, Loewen, and Vinter.

Perhaps the best known interval-varying problem is the time optimality problem of optimal control theory, namely, to steer a given system to the origin in minimal time. In a simple differential inclusion formulation, this is to find the least $T \geq 0$ such that a trajectory for E evolves from α $(\neq 0)$ to 0 as t goes from 0 to T:

$$\text{minimize } T: \ \dot{x}(t) \in E(x(t)), \ x(0) = \alpha, \ x(T) = 0. \tag{4.25}$$

We shall denote by $T(\alpha)$ the minimal time, and we suppose that E satisfies the same hypotheses as in §4.3.

Let us first set the stage with the props with which we have now become familiar. The necessary conditions for this problem [C1983] are in terms of an arc p satisfying (together with the solution x) the Hamiltonian inclusion $(-\dot{p}, \dot{x}) \in \partial H(x,p)$ almost everywhere on $[0,T]$ and the condition $H(x(t),p(t)) =$ constant $= h$. The arc $p = 0$ always satisfies these conditions when $h = 0$. If

this arc is the only one corresponding to $h = 0$, then the problem (25) is termed normal. The set of p satisfying these conditions for $h = 1$ is denoted $M(x)$. The notation $\theta M(\Sigma)$ designates the set

$$\{-p(0) : p \in M(x), x \in \Sigma\},$$

where Σ is the set of trajectories that solves the problem.

THEOREM 4.10 [CL1986]. *Let the minimal time problem* (4.25) *be normal, and let* Σ *be the set of trajectories that solves it. Then* T *is Lipschitz near* α*, and its subdifferential satisfies*

$$\partial T(\alpha) = \text{co}\{\theta M(\Sigma) \cap \partial T(\alpha)\}.$$

Many of us will have actually solved minimal time problems in courses on optimal control, especially in the linear case: $E(x) = \{Ax + Bu : u \in U\}$. Frequently these problems admit a unique solution x; and in computing x, the unique adjoint arc p of the maximum principle (the only element of $M(x)$ in the linear case) is calculated as well. The theorem shows that this arc p contains sensitivity information. In the case cited above, we have

$$T(\alpha + \beta) = T(\alpha) - \langle p(0), \beta \rangle + o(\beta).$$

What direction then is "towards the origin" relative to the flow? That is, in what direction should a small change in the initial position α be made to most reduce the steering time to 0? Clearly, the answer is $p(0)$. Incidentally, when $E(x)$ is of the form $\{Ax + Bu : u \in U\}$, where U contains a neighborhood of 0, normality is equivalent to the familiar hypothesis that the following controllability matrix have maximal rank:

$$[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B].$$

Another issue of interest concerns the behavior of T at the origin. Simple examples illustrate that at zero (in contrast to other points), Lipschitz behavior is not to be expected. Continuity is not always present either, but it is a more reasonable hope.

We can develop a verifiable criterion for the continuity of T at 0. Suppose that we have $0 \in E(0)$; then the arc $x(t) \equiv 0$ is a trajectory. We shall say "the origin is normal" provided that for all sufficiently small τ, the zero arc on $[0, \tau]$ is normal, in the sense defined above.[36]

THEOREM 4.11. *If the origin is normal, then* T *is continuous at* 0.

The proof of this theorem is an application of value function analysis. A certain auxiliary problem is defined and subjected to parameter perturbation. Normality is proven and shown via the subdifferential formula to imply Lipschitz behavior of the associated value function. This in turn implies continuity of T.

The minimal time problem admits the *final* time as a choice variable. Many interesting problems feature *intermediate* variable times whose determination is one of the principal features of their solution. And, in many cases, the underlying

[36] In the linear case mentioned above, normality of the origin is again equivalent to the controllability matrix's having full rank.

system undergoes a qualitative change at these intermediate times. A systematic theory of such problems, that of *multiprocesses*, has been developed to treat such optimal control processes [CV1989a], [CV1989b]. Here we shall describe only a special case of the theory, in particular limiting ourselves to certain two-stage multiprocesses.

The two-stage multiprocess that we now consider consists of the two processes $(x_1, u_1, f_1, F_1, U_1)$ and $(x_2, u_2, f_2, F_2, U_2)$ on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, each of the type (4.1)–(4.3), but defined on variable time intervals $[0, T_1]$ and $[T_1, T_2]$, respectively, where T_1 and T_2 are subject to choice. We seek to minimize

$$\int_0^{T_1} F_1(x_1(t), u_1(t))dt + \int_{T_1}^{T_2} F_2(x_2(t), u_2(t))dt.$$

The constraints on the endpoints and on T_1, T_2 are specified jointly in terms of a given closed set C:

$$(T_1, T_2, x_1(T_1), x_2(T_1)) \in C,$$
$$x_1(0) = x_0, \qquad x_2(T_2) = x_T.$$

The following necessary conditions of maximum principle type hold under the hypothesis that the data are locally Lipschitz in x. We denote by $\widetilde{H}_i(x, p, u)$ the pseudo-Hamiltonian $\langle p, f_i(x, u)\rangle - F_i(x, u)$, and for simplicity we state only the normal form of the conditions.

THEOREM 4.12. *If the multiprocess problem above is normal, then corresponding to a solution there exist arcs p_1, p_2 on $[0, T_1]$, $[T_1, T_2]$, respectively, and constants h_1, h_2 such that for $i = 1, 2$ (on the appropriate interval in each case)*

$$-\dot{p}_i \in \partial_x \widetilde{H}_i(x_i, p_i, u_i) \quad a.e., \tag{4.26}$$

$$\widetilde{H}_i(x_i, p_i, u_i) = \max_{w \in U_i} \widetilde{H}_i(x_i, p_i, w) = h_i \quad a.e., \tag{4.27}$$

$$(h_1 - h_2, h_2, -p_1(T_1), p_2(T_1)) \in N_C(T_1, T_2, x_1(T_1), x_2(T_1)). \tag{4.28}$$

Before looking at an example of the use of the theorem, let us consider how we might solve this two-stage problem without it. If the optimal T_1 is known, then the problem reduces to two standard ones. The first is on the fixed interval $[0, T_1]$, the second is on $[T_1, T_2]$, where T_2 is still to be determined and endpoint constraints are inherited from C. Suppose that this problem can be solved for each T_1, and the resulting value $V(T_1)$ calculated. Then it remains to minimize V over T_1, for which a necessary condition is $\nabla V(T_1) = 0$, or more realistically as we know, $0 \in \partial V(T_1)$.

All of this depends upon being able to develop a reasonably explicit expression for $V(T_1)$, and this is not always possible. The point of Theorem 4.12 is that it embodies this sort of "stationarity of an intermediate value function" without having to calculate it. The condition $0 \in \partial V(T_1)$ is part of (4.28), together with transversality conditions.

The relation (4.28) reduces to readily interpretable conditions in special cases, such as Snell's laws, when refraction and reflection are modeled or when T_1 is (locally) free, in which case (4.28) yields $h_1 = h_2$ so that the Hamiltonians are "continuous across the interface."

Example.[37] The problem consists of steering a robot arm from rest at the origin to an assigned point, where it drops a load and then returns to the origin. The objective is to minimize the time taken to accomplish the task. More precisely we have the following:

$$\begin{aligned}
&\text{minimize } T_2\\
&\text{subject to}\\
&\ddot{x} = \begin{cases} m_1^{-1}u, & 0 \le t < T_1,\\ m_2^{-1}u, & T_1 < t \le T_2,\end{cases}\\
&u(t) \in [-1,1],\\
&0 \le T_1 \le T_2, \qquad x(T_1) = \overline{x},\\
&x(0) = \dot{x}(0) = x(T_2) = \dot{x}(T_2) = 0.
\end{aligned}$$

Here the same unindexed notation (x, u, U) has been used for both parts of the multiprocess, which seems natural; u is the applied acceleration, x the (linear) position, m_1 and m_2 the mass of the robot before and after drop-off respectively ($m_2 < m_1$), T_1 the time of drop-off, $\overline{x}$ the (given) drop-off position, and T_2 the time of return to rest at the origin.

The problem is phrased in first-order terms in the usual way. The state equation for $(x, v) \in \mathbb{R}^2$ becomes, for $t < T_1$, for example,

$$\dot{x} = v, \qquad \dot{v} = m_1^{-1}u.$$

The functions F_1, F_2 for our problem are identically 1, and the set C is seen to be

$$\{(T_1, T_2, \overline{x}, \gamma, \overline{x}, \gamma) : \ 0 < T_1 < T_2, \gamma \in \mathbb{R}\}.$$

A straightforward analysis of the necessary conditions that result from the application of Theorem 4.12 leads to a formula specifying $\overline{v} := \dot{x}(T_1-) = \dot{x}(T_1+)$. (It turns out that $\overline{v}$ is positive, incidentally, so that the load is dropped while the arm is still moving away from the origin.) Armed with this value (which can be shown to correspond precisely to the condition $\nabla V(\overline{v}) = 0$ for an appropriate intermediate value function), we note that a phase plane analysis along standard lines leads to a synthesis of the solution. The point is that this information is available even when the intermediate value functions are not.[38] □

[37] This is a special case of one of the examples in [CV1989b].

[38] A further aspect of the theory that we have not presented is its ability to give complete necessary conditions for the first time for optimal control problems depending only measurably on the time variable. The key is once again proximal normal analysis [CLV1988].

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